SRI RAMAKRISHNA INSTITUTE OF TECHNOLOGY

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CALCULUS AND LINEAR ALGEBRA

Differential Calculus

Introduction:

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit as we know and understand it today did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach.

Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Notice that $x = 1$ does not belong to the domain of $f(x)$. Otherwise, we would like to know how $f(x)$ behaves close to the point $x = 1$.

We start with a table of values:

x	0.8	0.9	0.95	0.99	1.01	1.05	1.1
$f(x) = \frac{x^2-1}{x-1}$	1.8	1.9	1.95	1.99	2.01	2.05	2.1

It appears that for values of x close to 1 we have that $f(x)$ is close to 2. In fact, we can make the values of $f(x)$ as close to 2 as we like by taking x sufficiently close to 1. We express this by saying the limit of the function $f(x)$ as x approaches 1 is equal to 2 and use the notation:

$$\lim_{x \to 1} f(x) = 2$$

The limit of a function

The limit of a function at a point a in its domain (if it exists) is the value that the function approaches as its argument approaches $\boldsymbol{a}$.

Cauchy Definitions of Limit

Let $f(x)$ be a function that is defined on an open interval X containing $x = a$. (The value $f(a)$ need not be defined.)

The number L is called the limit of function $f(x)$ as $x \to a$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

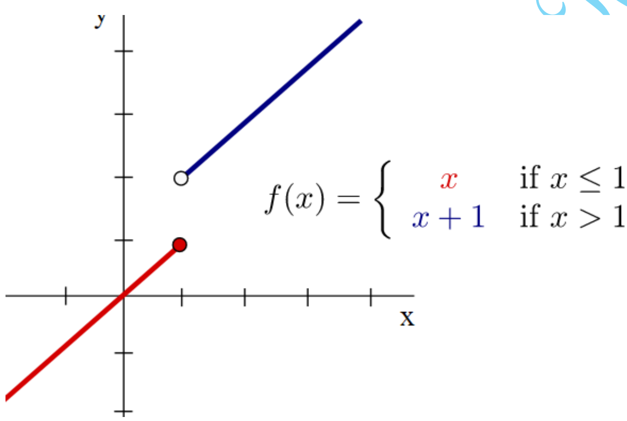
$$|f(x) - L| < \varepsilon$$

whenever

$$0 < |x - a| < \delta$$

One sided limits

Consider the following piecewise defined function:



Observe from the graph that as $\boldsymbol{x}$ gets closer and closer to 1 from the left, then $\boldsymbol{f(x)}$ approaches $+\mathbf{1}$. Similarly, as $\boldsymbol{x}$ gets closer and closer $\mathbf{1}$ from the right, then $\boldsymbol{f(x)}$ approaches $+\mathbf{2}$. We use the following notation to indicate this:

$$\lim_{x \to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2$$

The symbol $x \to 1^-$ means that we only consider values of x sufficiently close to 1 which are less than 1. Similarly, the symbol $x \to 1^+$ means that we only consider values of x sufficiently close to 1 which are greater than 1.

Left-hand limit

Let $\lim_{x \to a^-}$ denote the limit as x goes toward a by taking on values of x such that $x < a$. The corresponding limit $\lim_{x \to a^-} f(x)$ is called the left-hand limit of $f(x)$ at the point $x = a$.

Right - hand limit

Similarly, let $\lim_{x \to a^+}$ denote the limit as x goes toward a by taking on values of x such that $x > a$. The corresponding limit $\lim_{x \to a^+} f(x)$ is called the right-hand limit of $f(x)$ at $x = a$.

More over the two sided limit $\lim_{x \to a} f(x)$ exists only if both one-sided limits exist and are equal to each other, that is $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

In this case,

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

Problem: 1

Use a calculator to estimate $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)$, where x is in radians.

Answer:

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
3	0.04704	−3	0.04704
2	0.454649	−2	0.454649
1	0.841471	−1	0.841471
0.5	0.958851	−0.5	0.958851
0.25	0.989616	−0.25	0.989616
0.025	0.999896	−0.025	0.999896
0.0025	0.999999	−0.0025	0.999999
0.00025	1	−0.00025	1

From table we see that

- $x \to 0^+$, $f(x)$ appears to be approaching 1.
- $x \to 0^-$, $f(x)$ appears again to be approaching 1.

$$\therefore \quad \lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$$

Problem: 2

Use a calculator to estimate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$.

Answer:

x	$\frac{\sqrt{x} - 2}{x - 4}$
3.9	0.251582
3.99	0.250156
3.999	0.250016
3.9999	0.250002
4.0001	0.249998
4.001	0.249984
4.01	0.249844
4.1	0.248457

From table we see that

- $x \to 4$, $f(x)$ appears to be approaching 2.5.

$$\therefore \quad \lim_{x \to 4} \left(\frac{\sqrt{x} - 2}{x - 4}\right) = 2.5$$

Problem: 3

Use a calculator to estimate $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$.

Answer:

x	$\sin\left(\frac{1}{x}\right)$	x	$\sin\left(\frac{1}{x}\right)$
-0.1	0.544021	0.1	0.173648
-0.01	0.506366	0.01	-0.506366
-0.001	-0.826880	0.001	0.826800
-0.0001	0.305614	0.0001	-0.305614
-0.00001	-0.035749	0.00001	0.035749

We see that $\sin\left(\frac{1}{x}\right)$ oscillates wildly between -1 and 1 a s x approaches 0.

Problem: 4

For the function $f(x) = \begin{cases} x+1; & if\ x < 2 \\ x^2-4; & if\ \ x \geq 2 \end{cases}$ evaluate $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

Answer:

x	$f_1(x) = x+1$	x	$f_2(x) = x^2-4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001

From table we see that

- $x \to 2^+$, $f(x)$ appears to be approaching 0.
- $x \to 2^-$, $f(x)$ appears again to be approaching 3.

$$\therefore \lim_{x \to 2^-} f(x) = 3$$

$$\therefore \lim_{x \to 2^+} f(x) = 0$$

Problem: 5

Evaluate the limits $\lim_{x \to 2^-} \frac{|x^2-4|}{x-2}$ and $\lim_{x \to 2^+} \frac{|x^2-4|}{x-2}$.

Answer:

x	$f(x)=\dfrac{\lvert x^2-4\rvert}{x-2}$	x	$f(x)=\dfrac{\lvert x^2-4\rvert}{x-2}$
1.9	−3.9	2.1	4.1
1.99	−3.99	2.01	4.01
1.999	−3.999	2.001	4.001
1.9999	−3.9999	2.0001	4.0001

From table we see that

- $x \to 2^+$, $f(x)$ appears to be approaching 4.
- $x \to 2^-$, $f(x)$ appears again to be approaching −4.

$$\therefore \lim_{x\to 2^-} f(x) = -4$$

$$\therefore \lim_{x\to 2^+} f(x) = 4$$

Limits as x Approaches a Particular Number

Problem: 6

Find the limit as p approaches 10 of the expression k = 3p + 7.

Answer:

By definition of limit we write the given expression as

$$\lim_{p\to 10}[3p+7] = 3(10)+7$$

$$= 37$$

Problem: 7

We know that x cannot equal 5 in the following expression (because we cannot have a denominator equal to zero): $f(x)=\frac{x^2-8x+15}{x-5}$. What is the value of the function as x approaches 5?

Answer

By definition of limit we write the given expression as

$$\lim_{x\to 5} f(x) = \frac{x^2-8x+15}{x-5}$$

Factoring $f(x)$, we get

$$\lim_{x \to 5} f(x) = \lim_{x \to 5}\left[\frac{x^2 - 8x + 15}{x - 5}\right]$$

$$= \lim_{x \to 5}\left[\frac{(x-5)(x-3)}{x-5}\right]$$

$$= \lim_{x \to 5}[x - 3]$$

$$= 5 - 3$$

$$= 2$$

Limits as x Approaches 0

Problem: 8

Find the limit as x approaches 0 of $\frac{\sin x}{x}$.

Answer:

By definition of limit we write the given expression as

$$\lim_{x \to 0} f(x) = \frac{\sin x}{x}$$

By L Hospital rule, we have

$$\lim_{x \to 0} f(x) = \frac{\cos x}{1}$$

$$= \frac{\cos 0}{1}$$

$$= 1$$

Limits as x Approaches Infinity

Problem: 9

Find the limit of the function $f(x) = \frac{5}{x}$ as x approaches to ∞.

Answer

Clearly, if we take larger and larger values of x, the value of the fraction becomes smaller and smaller until it gets very close to 0. We say that "the limit of $\frac{5}{x}$ as x approaches infinity is 0.

$$i.e., \quad \lim_{x \to 0} \left(\frac{5}{x}\right) = 0$$

In general

$$\lim_{x \to 0} \left(\frac{1}{x^2}\right) = 0$$

$$\lim_{x \to 0} \left(\frac{1}{x^9}\right) = 0$$

Limits when the variable is in the denominator

Problem: 10

Given $f(x) = 4x^7 - 18\,x^3 + 9$, then find the following limits.

A. $\lim_{x \to -\infty} f(x)$ B. $\lim_{x \to \infty} f(x)$

Answer:

A.
$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} [4x^7 - 18\,x^3 + 9]$$
$$= \lim_{x \to -\infty} \left(x^7 \left[4 - \frac{18}{x^4} + \frac{9}{x^7}\right]\right)$$
$$= \lim_{x \to -\infty} x^7 \times \lim_{x \to -\infty} \left[4 - \frac{18}{x^4} + \frac{9}{x^7}\right]$$
$$= (-\infty)^7 \times \left[4 - \frac{18}{\infty} + \frac{9}{\infty}\right]$$
$$= -\infty \times [4 - 0 + 0]$$
$$= -\infty$$

B.
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} [4x^7 - 18\,x^3 + 9]$$
$$= \lim_{x \to \infty} \left(x^7 \left[4 - \frac{18}{x^4} + \frac{9}{x^7}\right]\right)$$
$$= \lim_{x \to \infty} x^7 \times \lim_{x \to \infty} \left[4 - \frac{18}{x^4} + \frac{9}{x^7}\right]$$
$$= (\infty)^7 \times \left[4 - \frac{18}{\infty} + \frac{9}{\infty}\right]$$
$$= \infty \times [4 - 0 + 0]$$
$$= \infty$$

Problem: 11

Evaluate $\lim_{x \to \infty} \frac{2x+3}{2-x}$

Answer:

$$\lim_{x \to \infty} \frac{2x+3}{2-x} = \lim_{x \to \infty} \frac{x\left(2+\frac{3}{x}\right)}{x\left(\frac{2}{x}-1\right)}$$

$$= \lim_{x \to \infty} \frac{\left(2+\frac{3}{x}\right)}{\left(\frac{2}{x}-1\right)}$$

$$= \frac{2+\frac{3}{\infty}}{\frac{2}{\infty}-1} \qquad \left[\because \frac{1}{\infty} = 0\right]$$

$$= \frac{2+0}{0-1}$$

$$= -2$$

Problem: 12

Evaluate $\lim_{x \to \infty} \frac{2x-5x^3}{x^3-x+1}$

Answer:

$$\lim_{x \to \infty} \frac{2x-5x^3}{x^3-x+1} = \lim_{x \to \infty} \frac{x^3\left(\frac{2}{x^2}-5\right)}{x^3\left(1-\frac{1}{x^2}+\frac{1}{x^3}\right)}$$

$$= \lim_{x \to \infty} \frac{\frac{2}{x^2}-5}{1-\frac{1}{x^2}+\frac{1}{x^3}}$$

$$= \frac{\frac{2}{\infty}-5}{1-\frac{1}{\infty}+\frac{1}{\infty}}$$

$$= \frac{0-5}{1-0+0}$$

$$= -5$$

Problem: 13

Evaluate $\lim_{x \to \infty} \dfrac{5x^4 - 2x}{6x^3 + 7x^5 - 3}$

Answer:

$$\lim_{x \to \infty} \frac{5x^4 - 2x}{6x^3 + 7x^5 - 3} = \lim_{x \to \infty} \frac{x^4\left(5 - \frac{2}{x^3}\right)}{x^5\left(\frac{6}{x^2} + 7 - \frac{3}{x^5}\right)}$$

$$= \lim_{x \to \infty} \frac{\left(5 - \frac{2}{x^3}\right)}{x\left(\frac{6}{x^2} + 7 - \frac{3}{x^5}\right)}$$

$$= \frac{\left(5 - \frac{2}{\infty}\right)}{\infty\left(\frac{6}{\infty} + 7 - \frac{3}{\infty}\right)}$$

$$= \frac{5}{\infty}$$

$$= 0$$

Problem: 14

Evaluate $\lim_{x \to \infty} \dfrac{\sqrt{4x^2 - 2x}}{x + 3}$.

Answer:

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 - 2x}}{x + 3} = \lim_{x \to \infty} \frac{\sqrt{x^2\left(4 - \frac{2}{x}\right)}}{x\left(1 + \frac{3}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{4 - \frac{2}{x}}}{x\left(1 + \frac{3}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{x\sqrt{4 - \frac{2}{x}}}{x\left(1 + \frac{3}{x}\right)}$$

$$= \frac{\sqrt{4 - \frac{2}{\infty}}}{1 + \frac{3}{\infty}}$$

$$= \sqrt{4}$$

$$= 2$$

Continuity

Definition: 1

A function f is continouos at a number $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Remember that $\lim_{x \to a} f(x)$ describes both what is happening when x is slightly less than a and what is happening when x is slightly greater than a. Thus there are three conditions inherent in this definition of continuity. A function is continuous at a if the limit as $x \to a$ exists, and $f(a)$ exists, and this limit is equal to $f(a)$. This means that the following three values are equal:

$$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$$

That is the value as x approaches a from the left is the same as the value as x approaches a from the left (the limit exists) which is the same as the value of f at a

If any of these quantities is different, or if any of them fails to exist, then we say that $f(x)$ is discontinuous at $x = a$, or that $f(x)$ has a discontinuity at $x = a$.

Definition: 2

- A function "f" is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
- A function "f" is said to be continuous in a closed interval $[a, b]$ if
  i. f is continuous in (a, b)
  ii. $\lim_{x \to a^+} f(x) = f(a)$
  iii. $\lim_{x \to a^-} f(x) = f(a)$

Discontinuity Definition

The function "f" will be discontinuous at $x = a$ a in any of the following cases:

- $f(a)$ is not defined
- $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist but are not equal $f(a)$ in a given interval

Types of discontinuity

There are four types of discontinuities as follows.

- Jump Discontinuity
- Point Discontinuity
- Essential Discontinuity or Infinite Discontinuity
- Removable Discontinuity

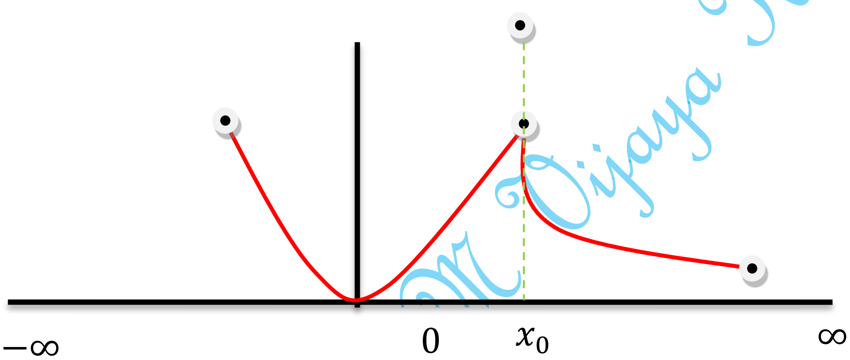
I. Jump Discontinuity

It has both one-sided limits exist, but have different values.

A real-valued function $f(x)$ has a jump discontinuity at a point x_0 in its domain provided that

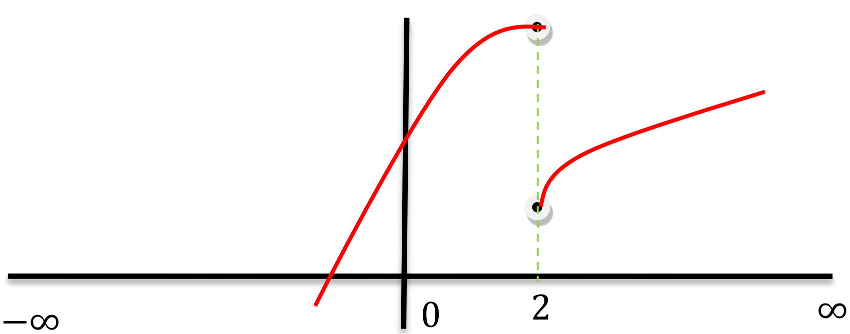
$$\lim_{x \to x_0^-} f(x) = L_1; \quad \text{and} \quad \lim_{x \to x_0^+} f(x) = L_2$$

both exist but $L_1 \neq L_2$.



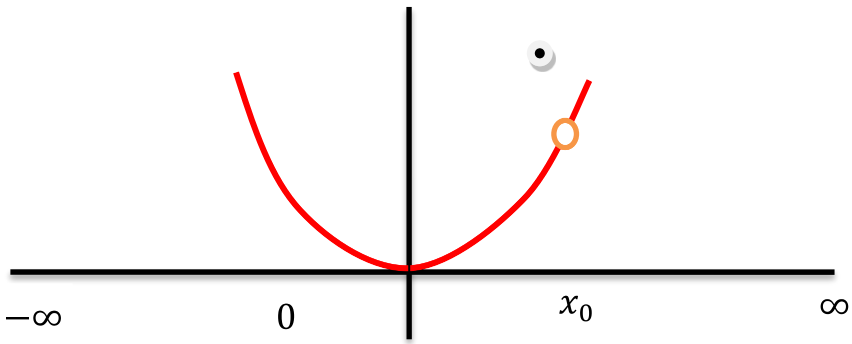
Example:

Consider the function $f(x) = \begin{cases} 5 - x^2; & x < 2 \\ \sqrt{x}; & x \geq 2 \end{cases}$.



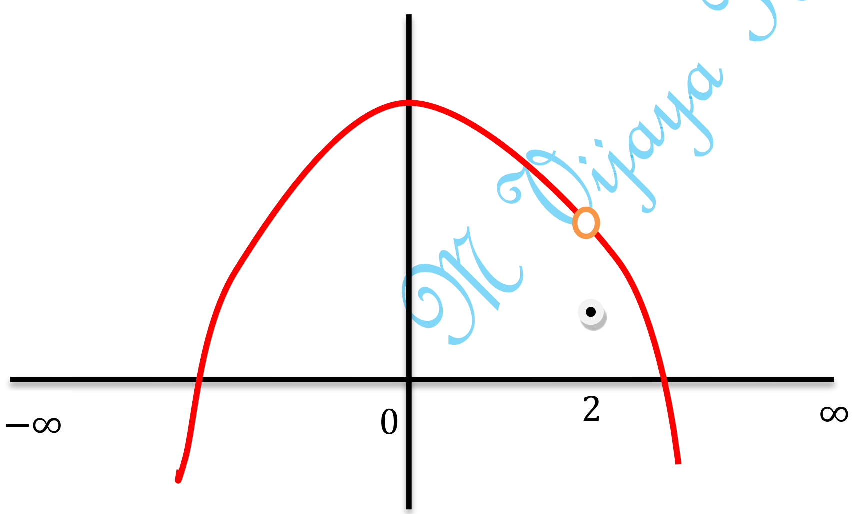
II. Point discontinuity

Point discontinuities exist for piecewise functions where a specific value for $\boldsymbol{x_0}$ is defined differently than the rest of the piecewise function.



Example:

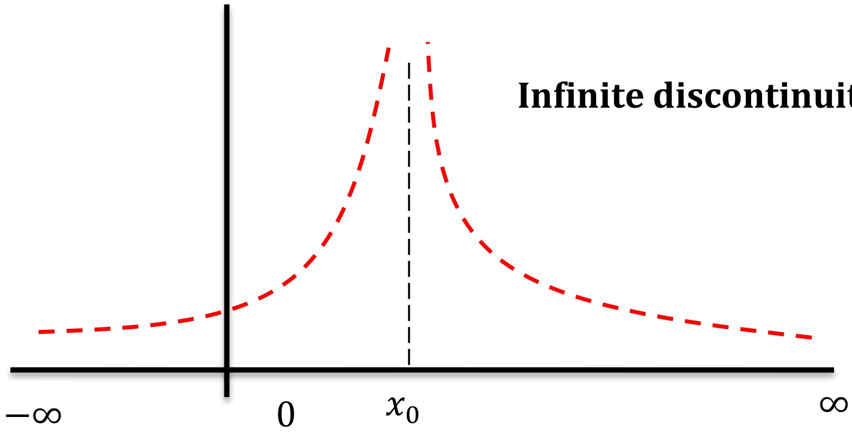
Consider the function $f(x) = \begin{cases} 2 - x^2; & x \neq 2 \\ 1; & x = 2 \end{cases}$



III. Essential or Infinite Discontinuities:

It has both right sides and left sided limits are infinite.

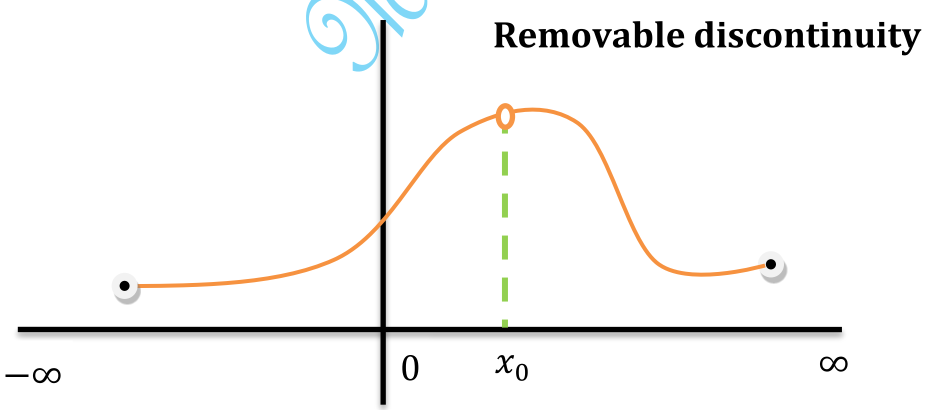
A real-valued function $\boldsymbol{f(x)}$ is said to have an infinite discontinuity at a point $\boldsymbol{x_0}$ in its domain provided that either or both the lower or upper limits fails to exist as $\boldsymbol{x}$ tends to $\boldsymbol{x_0}$.



III. Removable Discontinuity:

A removable discontinuity is a point on the graph that is undefined or does not fit the rest of the graph. There is a gap at that location when you are looking at the graph. When graphed, a removable discontinuity is marked by an open circle on the graph at the point where the graph is undefined or is a different value like this.

A real-valued function $\boldsymbol{f(x)}$ is said to have a removable discontinuity at a point $\boldsymbol{x_0}$ in its domain provided that both $\boldsymbol{f(x_0)}$ and $\lim\limits_{x \to x_0} f(x) = L < \infty$ exist while $\boldsymbol{f(x_0) \neq L}$.



Example:

Consider the function $f(x) = \dfrac{x^2 - 2x}{x^2 - 4}$.

This function has a removable discontinuity at $x = 2$.

Now we redefine this function as follows

$$f(x) = \frac{x(x-2)}{(x+2)(x-2)} = \frac{x}{x+2}$$

Now $\lim_{x \to 2} f(x) = \lim_{x \to 2} \left[\frac{x}{x+2}\right] = \frac{2}{2+2} = \frac{1}{2}$

$$\therefore \quad f(x) = \begin{cases} \frac{x^2-2x}{x^2-4}; & x \neq 2 \\ \frac{1}{2}; & x = 2 \end{cases}$$

Intermediate Value Theorem (IVT)

Let $f(x)$ be continuous on the interval $[a,b]$ with $f(a) = A$ and $f(b) = B$. Given any value C between A and B, there is at least one point $c \in [a,b]$ with $f(c) = C$. Where $A,\ B$ and are some constants.

Remark

Suppose that $f(x)$ is continuous on the interval $[a,b]$ with $f(a) < 0$ and $f(b) > 0$. Then there is a point $c \in [a,b]$ such that $f(c) = 0$.

Problem: 1

Show that the equation $x^3 - 3x^2 + 1 = 0$ has a solution on the interval $(0,1)$.

Answer:

$f(x) = x^3 - 3x^2 + 1$ is continuous in $(0,1)$

$f(0) = 0 - 0 + 1 = 1 > 0$

$f(1) = 1 - 3 + 1 = -1 < 0$

Hence by the IVT, there is a $c \in [a,b]$ with $f(c) = 0$.

Problem: 2

Show that the equation $3x^5 - 4x^2 - 3 = 0$ has a solution on the interval $(0,2)$.

Answer:

$f(x) = 3x^5 - 4x^2 - 3$ is continuous in $(0,2)$

$f(0) = 3(0) - 4(0) - 3 = -3 < 0$

$f(2) = 3(2^5) - 4(2^2) - 3 = 77 > 0$

Hence by the IVT, there is a $c \in [a,b]$ with $f(c) = 0$.

Problem: 3

Use the Intermediate Value Theorem to prove that the equation $x^3 + 2x - 5 = 0$ is solvable.

Answer:

First we need to find the interval (a, b).

Given $\quad f(x) = x^3 + 2x - 5 \quad \dots (1)$

$f(0) = 0 + 2(0) - 5 = -5 \quad [-\text{ve}]$

$f(1) = 1^3 + 2(1) - 5 = -3 \quad [-\text{ve}]$

$f(2) = 2^3 + 2(2) - 5 = +7 \quad [+\text{ve}]$

$\therefore$ The function $f(x)$ is continuous in $(1,2)$.

Hence by the IVT, there is a $c \in [a, b]$ with $f(c) = 0$.

Therefore given equation $x^3 + x - 5 = 0$ is solvable.

Problem: 4

Use the IVT theorem to prove that the equation $x^3 - \sqrt{x} - 20 = 0$ is solvable.

Answer:

First we need to find the interval (a, b).

Given $\quad f(x) = x^3 - \sqrt{x} - 20 \quad \dots (1)$

$f(0) = 0 - 0 - 20 = -20 \quad [-\text{ve}]$

$f(1) = 1^3 - \sqrt{1} - 20 = -18 \quad [-\text{ve}]$

$f(2) = 2^3 - \sqrt{2} - 20 = -13.41 \quad [-\text{ve}]$

$f(3) = 3^3 - \sqrt{3} - 20 = -12.73 \quad [-\text{ve}]$

$f(4) = 4^3 - \sqrt{4} - 20 = 42 \quad [+\text{ve}]$

$\therefore$ The function $f(x)$ is continuous in $(3,4)$.

Hence by the IVT, there is a $c \in [a, b]$ with $f(c) = 0$.

Therefore given equation $x^3 - \sqrt{x} - 20 = 0$ is solvable.

Derivative

The derivative of a function is one of the basic concepts of mathematics. Together with the integral, derivative occupies a central place in calculus. The process of finding the derivative is called differentiation. The inverse operation for differentiation is called integration.

The derivative of a function at some point characterizes the rate of change of the function at this point. We can estimate the rate of change by calculating the ratio of change of the function Δy to the change of the independent variable Δx. In the definition of derivative, this ratio is considered in the limit as $\Delta x \to 0$.

Definition:

Let $f(x)$ be a function whose domain contains an open interval about some point x_0. Then the function $f(x)$ is said to be differentiable at x, and the derivative of $f(x)$ at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Lagrange's notation (Derivative of the function)

$y = f(x)$ as $f'(x)$ or $y'(x)$

Leibniz's notation (Derivative of the function)

$y = f(x)$ as $\frac{df}{dx}$ or $\frac{dy}{dx}$

Steps to find the derivative of a function

1. Form the difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
2. Simplify the quotient, canceling Δx if possible
3. Find the derivative $f'(x_0)$, applying the limit to the quotient. If this limit exists, then we say that the function $f(x)$ is differentiable at x_0

Basic Differentiation Rules

Derivative of a Constant

If $f(x) = C$, then its derivative $f'(x) = 0$.

Constant Multiple Rule

Let k be a constant. If $f(x)$ is differentiable, then $k\,f(x)$ is also differentiable and then its derivative defined by

$$[k\,f(x)]' = k\,f'(x)$$

Sum Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions. Then the sum of two functions is also differentiable and then its differentiation is defined as

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

In general we may consider n functions namely $f_1(x),\ f_2(x), \dots . f_n(x)$ be differentiable. Then their sum is also differentiable and its derivative is

$$[f_1(x) + f_2(x) + \cdots + f_n(x)]' = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$$

Difference Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions. Then the difference of two functions is also differentiable and then its differentiation is defined as

$$[f(x) - g(x)]' = f'(x) - g'(x)$$

In general we may consider n functions namely $f_1(x),\ f_2(x), \dots . f_n(x)$ be differentiable. Then their difference is also differentiable and its derivative is

$$[f_1(x) - f_2(x) - \cdots - f_n(x)]' = f_1'(x) - f_2'(x) - \cdots - f_n'(x)$$

Linear Combination Rule

Suppose $f(x)$ and $g(x)$ are differentiable functions and a and b are real numbers. Then the function $h(x) = a\,f(x) + b\,g(x)$ is also differentiable and $h'(x) = a\,f'(x) + b\,g'(x)$.

Product rule

The product rule is a formal rule for differentiating problems where one function is multiplied by another.

Suppose $u(x)$and $v(x)$ be any two differentiable functions. Then the product of the functions $u(x)v(x)$ is also differentiable and we define

$$\frac{d}{dx}[u(x) \times v(x)] = u(x) \times \frac{d}{dx}[v(x)] + v(x) \times \frac{d}{dx}[u(x)]$$

or $$d(uv) = uv' + vu'$$

Quotient rule

Suppose $u(x)$and $v(x)$ be any two differentiable functions provided $v(x) \neq 0$. Then we differentiate the functions $u(x)/v(x)$ is also differentiable and we define

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\,u' - u\,v'}{v^2}; \quad v \neq 0$$

The Chain rule

The chain rule states that the derivative of $f[g(x)]$.

In other words, it helps us differentiate *composite functions*.

Eg:

$$f[g(x)] = \ln[\sin x]$$

Here $f(x) = \ln[\sin x]$ and $g(x) = \sin x$

Explicit Functions:

When a function is written so that the dependent variable is isolated on one side of the equation, we call it an explicit function.

In simple words we say that a function defined in the form $y = f(x)$ is said to be "explicitly" defined.

Eg:

$y = x^2;$

$y = ax^2 + bx + c$

Implicit Functions:

When the dependent variable is not isolated, we refer to the equation as being implicitly defined and the y -variable as an implicit function.

Eg:

$x^2 + y^2 = a^2$;

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Derivatives of basic elementary functions

Constant function

Problem: 1

Show that the derivative of a constant is zero by using the definition of derivative.

Answer:

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (1)$$

Consider the function $f(x)$ is always equal to a constant say C.

$\therefore \quad \boldsymbol{f(x) = c}$

and $\boldsymbol{f(x+\Delta x) = c}$.

Using in equation (1), we get

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{c - c}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{0}{\Delta x}\right]$$

$$= 0$$

$\therefore \boldsymbol{f'(c) = 0.}$

Problem: 2

Find the derivative of a function x by using the definition of derivative.

Answer:

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (1)$$

Consider the function $\boldsymbol{f(x) = x}$.

and $\boldsymbol{f(x+\Delta x) = x + \Delta x}$.

Using in equation (1), we get

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x) - x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} [1]$$

$$= 1$$

$\therefore$ $\boldsymbol{f'(x) = 1.}$

Problem: 3

Find the derivative of a quadratic function $y = Ax^2 + Bx + C$ by using the definition of derivative, where A, B and C are some constants.

Answer:

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (1)$$

Consider the function $\boldsymbol{f(x) = Ax^2 + Bx + C}$.

$$\therefore \quad f(x+\Delta x) = [A(x+\Delta x)^2 + B(x+\Delta x) + C\,].$$

$$= A(x^2 + (\Delta x)^2 + 2x\,\Delta x) + Bx + B\,\Delta x + C.$$

$$= Ax^2 + A(\Delta x)^2 + 2Ax\,\Delta x + Bx + B\,\Delta x + C.$$

$$f(x+\Delta x) - f(x) = Ax^2 + A(\Delta x)^2 + 2Ax\,\Delta x + Bx + B\,\Delta x + C - Ax^2 - Bx - C$$

$$= A(\Delta x)^2 + 2Ax\,\Delta x + B\,\Delta x$$

$$\boldsymbol{f(x+\Delta x) - f(x) = \Delta x[A\Delta x + 2Ax + B\,]} \qquad \dots \mathbf{(2)}$$

Using in equation (1), we get

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta x[A\Delta x + 2Ax + B\,]}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \,[A\Delta x + 2Ax + B\,]$$

$$= 0 + 2Ax + B$$

$$= 2Ax + B$$

$$\therefore\; \boldsymbol{f'(Ax^2 + Bx + c) = 2Ax + B.}$$

Problem: 4

Find the derivative of a function $y = x^n$ by using the definition of derivative.

Answer:

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \dots \mathbf{(1)}$$

Consider the function $f(x) = x^n$.

$$\therefore \quad f(x+\Delta x) = (x+\Delta x)^n.$$

We know that the expansion

$$(x+a)^n = x^n + nC_1x^{n-1}a + nC_2x^{n-2}a^2 + \cdots \quad + a^n;\;\; n > 1$$

$$(x+\Delta x)^n = x^n + nC_1x^{n-1}(\Delta x) + nC_2x^{n-2}(\Delta x)^2 + \cdots \quad + (\Delta x)^n$$

$$\therefore \quad f(\boldsymbol{x}+\boldsymbol{\Delta x}) - f(\boldsymbol{x}) = x^n + nC_1x^{n-1}(\Delta x) + nC_2x^{n-2}(\Delta x)^2 + \cdots \quad + (\Delta x)^n - x^n$$

$$= nC_1x^{n-1}(\Delta x) + nC_2x^{n-2}(\Delta x)^2 + \cdots \quad + (\Delta x)^n \qquad \dots \mathbf{(2)}$$

Hence equation (1) be written as

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{nC_1x^{n-1}(\Delta x) + nC_2x^{n-2}(\Delta x)^2 + \cdots \quad + (\Delta x)^n}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta x(nC_1 x^{n-1}(\Delta x) + nC_2 x^{n-2}(\Delta x)^2 + \cdots \quad + (\Delta x)^n)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} [nC_1 x^{n-1} + nC_2 x^{n-2}(\Delta x) + \cdots \quad + (\Delta x)^{n-1}]$$

$$= nC_1 x^{n-1} + 0 + 0 + \cdots \quad + 0$$

$$= nx^{n-1}$$

$$\therefore \quad \boldsymbol{f'(x^n) = nx^{n-1}}.$$

Remarks:

- Suppose $n = 5$, then we have

  $f'(x^5) = 5x^{5-1} = 5x^4$
- Suppose $n = -1$, then we have

  $f'(x^{-1}) = -1x^{-1-1} = -1x^{-2} = \frac{-1}{x^2}$

Problem: 5

Find the derivative of a nature logarithmic function x by using the definition of derivative.

Answer:

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (\mathbf{1})$$

The natural logarithm of a function x is denoted by $\ln x$ or $\ln x$ or $\log_e x$.

Consider the function $\boldsymbol{f(x) = \ln x}$.

$$\therefore \quad \boldsymbol{f(x+\Delta x) = \ln(x+\Delta x)}.$$

$$\therefore \quad f(x+\Delta x) - f(x) = \ln(x+\Delta x) - \log x$$

$$= \ln\left(\frac{x+\Delta x}{x}\right) \qquad \left[\because \ \log A - \log B = \log\left(\frac{A}{B}\right)\right]$$

$$f(x+\Delta x) - f(x) = \ln\left(1 + \frac{\Delta x}{x}\right) \qquad \ldots (2)$$

Hence equation (1) be written as

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right)\right] \quad \dots (3)$$

Let us introduce the new variable say $n = \frac{x}{\Delta x}$; and $\Delta x = \frac{x}{n}$.

Subsequently we see that as $n \to \infty$ then $\Delta x \to 0$.

Now equation (3) be written as

$$f'(x) = \lim_{n \to \infty} \left[\frac{n}{x} \ln\left(1 + \frac{1}{n}\right)\right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{x} \ln\left(1 + \frac{1}{n}\right)^n\right] \quad [\because \log A^B = B \log A]$$

$$= \frac{1}{x} \ln\left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right] \quad \left[\because \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{x} \ln(e) \quad [\because \ln e = 1]$$

$$= \frac{1}{x}$$

$\therefore \quad \boldsymbol{f'(\ln x) = \frac{1}{x}}.$

Problem: 6

Find the derivative of an exponential function e^x by using the definition of derivative.

Answer:

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}} \quad \dots (1)$$

The exponential function of x is denoted by e^x.

Consider the function $\boldsymbol{f(x) = e^x}$.

$$\therefore \quad \boldsymbol{f(x + \Delta x) = e^{(x+\Delta x)}}.$$

$$\therefore \quad f(x + \Delta x) - f(x) = e^{(x+\Delta x)} - e^x$$

$$= e^x \times e^{\Delta x} - e^x \quad [\because A^{m\times n} = A^m \times A^n]$$

$$\boldsymbol{f(x + \Delta x) - f(x) = e^x(e^{\Delta x} - 1)} \quad \dots (2)$$

Hence equation (1) be written as

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} e^x (e^{\Delta x} - 1)\right]$$

$$= e^x \lim_{\Delta x \to 0} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right] \qquad \ldots (3)$$

To find the limit $\lim_{\Delta x \to 0} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right]$:

Let us introduce the new variable say $\boldsymbol{n = e^{\Delta x} - 1}$;

and $\lim_{\Delta x \to 0} n = e^0 - 1 = 1 - 1 = 0.$

Also $e^{\Delta x} = 1 + n$

Taking log on both sides, we get

$$\ln e^{\Delta x} = \ln(1 + n)$$

$$\Delta x = \ln(1 + n) \qquad [\because \ln e = 1]$$

Now equation (3) be written as

$$f'(x) = e^x \lim_{n \to 0} \left[\frac{n}{\ln(1 + n)}\right]$$

$$= e^x \lim_{n \to 0} \left[\frac{1}{\left(\frac{1}{n}\right) \ln(1 + n)}\right] \qquad \left[\because \; A = \frac{1}{\left(\frac{1}{A}\right)}\right]$$

$$= e^x \lim_{n \to 0} \left[\frac{1}{\ln(1 + n)^{\frac{1}{n}}}\right] \qquad [\because \; \ln A^B = B \ln A]$$

$$= e^x \left[\frac{1}{\ln\left\{\lim_{n \to 0} (1 + n)^{\frac{1}{n}}\right\}}\right] \qquad \left[\because \; \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e\right]$$

$$= e^x \left[\frac{1}{\ln e}\right] \qquad [\because \; \log e = 1]$$

$$= e^x$$

$\therefore \quad \boldsymbol{f'(e^x) = e^x}.$

Problem: 7

Find the derivative of a square root of function $\sqrt{x}$ using the definition of derivative.

Answer:

Consider the function $f(x) = \sqrt{x}$.

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (1)$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right]$$

Multiply and divide by $\sqrt{x+\Delta x} - \sqrt{x}$, we get

$$\boldsymbol{f'(x)} = \lim_{\Delta x \to 0} \left[\left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right) \times \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right)\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\left(\sqrt{x+\Delta x}\right)^2 - \left(\sqrt{x}\right)^2}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}\right] \qquad [\because \ (A-B)(A+B) = A^2 - B^2]$$

$$= \lim_{\Delta x \to 0} \left[\frac{x + \Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}\right]$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\therefore \ \boldsymbol{f'\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}}.$$

Problem: 8

Find the derivative of a reciprocal of a function x using the definition of derivative.

Answer:

Consider the function $f(x) = \frac{1}{x}$

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (1)$$

$$= \lim_{\Delta x \to 0} \left[\frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \right]$$

Cross multiplying quotient, we get

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} \left(\frac{x - (x+\Delta x)}{x(x+\Delta x)} \right) \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} \left(\frac{-\Delta x}{x^2 + x\Delta x} \right) \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{-1}{x^2 + x\Delta x} \right]$$

$$= \frac{-1}{x^2 + 0}$$

$$= -\frac{1}{x^2}$$

$$\therefore \quad \boldsymbol{f'\left(\frac{1}{x}\right) = \frac{-1}{x^2}}.$$

Problem: 9

Find the derivative of a trigonometric function $\sin x$ by using the definition of derivative.

Solution:

Consider the function $f(x) = \sin x$.

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \qquad \ldots (1)$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sin(x + \Delta x) - \sin x}{\Delta x}\right] \qquad \left[\because \ \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{2\cos\left(\frac{x + \Delta x + x}{2}\right) \times \sin\left(\frac{x + \Delta x - x}{2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cos\left(\frac{2x + \Delta x}{2}\right) \times \sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right] \qquad \left[\because \ A = \frac{1}{\left(\frac{1}{A}\right)}\right]$$

$$= \lim_{\Delta x \to 0} \left[\cos\left(x + \frac{\Delta x}{2}\right)\right] \times \lim_{\Delta x \to 0} \left[\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right]$$

$$= \cos(x + 0) \times 1 \qquad \left[\because \ \lim_{\Delta x \to 0} \left(\frac{\sin x}{x}\right)\right]$$

$$= \cos x$$

$\therefore$ $\boldsymbol{f'(\sin x) = \cos x}$

Problem: 10

Find the derivative of a trigonometric function $\cos x$ by using the definition of derivative.

Answer:

Consider the function $f(x) = \cos x$.

By definition of function derivative, we have

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}} \qquad \ldots (\mathbf{1})$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cos(x + \Delta x) - \cos x}{\Delta x}\right] \qquad \left[\because \ \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{-2\sin\left(\frac{x + \Delta x + x}{2}\right) \times \sin\left(\frac{x + \Delta x - x}{2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{-\sin\left(\frac{2x + \Delta x}{2}\right) \times \sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right] \qquad \left[\because \ A = \frac{1}{\left(\frac{1}{A}\right)}\right]$$

$$= \lim_{\Delta x \to 0} \left[-\sin\left(x + \frac{\Delta x}{2}\right)\right] \times \lim_{\Delta x \to 0} \left[\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right]$$

$$= -\sin(x + 0) \times 1 \qquad \left[\because \lim_{\Delta x \to 0} \left(\frac{\sin x}{x}\right) = 1\right]$$

$$= -\sin x$$

$\therefore$ $\boldsymbol{f'(\cos x) = -\sin x}$

Problem: 11

Find the derivative of a trigonometric function $\tan x$ by using the definition of derivative.

Answer:

Consider the function $f(x) = \tan x$.

$$f(x) = \frac{\sin x}{\cos x} \qquad \ldots (1)$$

Differentiating, we get

$$f'(x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \qquad \left[\because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{\cos x\,[\cos x] - \sin x\,[-\sin x]}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$\therefore$ $\boldsymbol{f'(\tan x) = \sec^2 x}$

Problem: 12

Find the derivative of a trigonometric function $\sec x$ by using the definition of derivative.

Answer:

Take $f(x) = \sec x$.

$$= \frac{1}{\cos x} \qquad \left[\because A^{-1} = \frac{1}{A}\right]$$

$$f(x) = [\cos x]^{-1} \qquad \dots (1)$$

Differentiating, we get

$$f'(x) = \frac{d}{dx}(\cos x)^{-1}$$

$$= -1(\cos x)^{-1-1}\frac{d}{dx}(\cos x)$$

$$= -(\cos x)^{-2}[-\sin x] \qquad \left[\because A^{-1} = \frac{1}{A}\right]$$

$$= \frac{\sin x}{(\cos x)^2}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$$= \sec x \times \tan x$$

$\therefore$ $\boldsymbol{f'(\sec x) = \sec x \tan x}$

Problem: 13

Find the derivative of a trigonometric function $\csc x$ by using the definition of derivative.

Answer:

Take $f(x) = \csc x$.

$$= \frac{1}{\sin x} \qquad \left[\because A^{-1} = \frac{1}{A}\right]$$

$$f(x) = [\sin x]^{-1} \qquad \dots (1)$$

Differentiating, we get

$$f'(x) = \frac{d}{dx}(\sin x)^{-1}$$

$$= -1(\sin x)^{-1-1}\frac{d}{dx}(\sin x)$$

$$= -(\sin x)^{-2}[\cos x] \qquad \left[\because A^{-1} = \frac{1}{A}\right]$$

$$= -\frac{\cos x}{(\sin x)^2}$$

$$= \frac{-1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$= -\csc x \times \cot x$$

$\therefore$ $\boldsymbol{f'(\csc x) = -\csc x \cot x}$.

Problem: 14

Find the derivative of a trigonometric function $\cot x$ by using the definition of derivative.

Answer:

Consider the function $f(x) = \cot x$.

$$f(x) = \frac{\cos x}{\sin x} \qquad \ldots (1)$$

Differentiating, we get

$$f'(x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \qquad \left[\because\ d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{\sin x\,[-\sin x] - \cos x\,[\cos x]}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-[\sin^2 x + \cos^2 x]}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

$\therefore$ $\boldsymbol{f'(\cot x) = -\csc^2 x}$.

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Problem: 15

Find the derivative of an inverse trigonometric function $\sin^{-1} x$.

Answer:

Given $y = sin^{-1} x$

$$\sin y = x; \qquad \left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$$

Differentiate using implicit method, we get

$$\cos y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \qquad \left[\because \cos\theta = \sqrt{1-\sin^2\theta}\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad [\because x = \sin y]$$

$$\therefore \quad \boldsymbol{\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

Problem: 16

Find the derivative of an inverse trigonometric function $\cos^{-1} x$.

Answer:

Given $y = \cos^{-1} x$

$$\cos y = x; \qquad \left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$$

Differentiate using implicit method, we get

$$-\sin y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\cos^2 y}} \qquad \left[\because \sin\theta = \sqrt{1-\cos^2\theta}\right]$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad [\because x = \cos y]$$

$$\therefore \quad \boldsymbol{\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

Problem: 17

Find the derivative of an inverse trigonometric function $\tan^{-1} x$.

Answer:

Given $y = \tan^{-1} x$

$\tan y = x;$

Differentiate using implicit method, we get

$$\sec^2 y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} \qquad [\because \sec^2\theta = 1 + \tan^2\theta]$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2} \qquad [\because x = \tan y]$$

$$\therefore \quad \boldsymbol{\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1 + x^2}}$$

Problem: 18

Find the derivative of an inverse trigonometric function $\cot^{-1} x$.

Answer:

Given $y = \cot^{-1} x$

$\cot y = x;$

Differentiate using implicit method, we get

$$-\csc^2 y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\csc^2 y}$$

$$\frac{dy}{dx} = \frac{-1}{1 + \cot^2 y} \qquad [\because \csc^2\theta = 1 + \cot^2\theta]$$

$$\frac{dy}{dx} = \frac{-1}{1 + x^2} \qquad [\because x = \cot y]$$

$$\therefore \quad \boldsymbol{\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1 + x^2}}$$

Problem: 19

Find the derivative of an inverse trigonometric function $\sec^{-1} x$.

Answer:

Given $y = \sec^{-1} x$

$\sec y = x;$

Differentiate using implicit method, we get

$$\sec y \tan y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \sqrt{\tan^2 y}} \qquad [\because \tan^2\theta = \sec^2\theta - 1]$$

$$\frac{dy}{dx} = \frac{1}{\sec y \sqrt{\sec^2\theta - 1}} \qquad [\because x = \sec y]$$

$$\boldsymbol{\frac{d(\cot^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2 - 1}}}$$

Problem: 20

Find the derivative of an inverse trigonometric function $\csc^{-1} x$.

Answer:

Given $y = \csc^{-1} x$

$\csc y = x;$

Differentiate using implicit method, we get

$$-\csc x \cot x \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\csc y \cot y}$$

$$\frac{dy}{dx} = \frac{-1}{\csc y \sqrt{\cot^2 y}} \qquad [\because \cot^2\theta = \csc^2\theta - 1]$$

$$\frac{dy}{dx} = \frac{-1}{\csc y \sqrt{\csc^2\theta - 1}} \qquad [\because x = \csc y]$$

$$\therefore \quad \boldsymbol{\frac{d(\csc^{-1} x)}{dx} = \frac{-1}{x\sqrt{x^2 - 1}}}$$

Problems based on differentiation:

Problem: 1

Differentiate the following:

i. $y = 4x^{\frac{3}{5}}$; ii. $y = \frac{1}{\sqrt{x^3}}$ iii. $y = x^2\sqrt[3]{x^5}$

iv. $y = \frac{x-3}{\sqrt{x}}$; v. $y = \left(x + \frac{1}{x}\right)^2$ vi. $y = \frac{(2x+3)(x+3)}{x^2}$

Answer: i

Given $y = 4x^{\frac{3}{5}}$

Differentiate, we get

$$y' = \frac{dy}{dx} = 4\frac{d}{dx}\left(x^{\frac{3}{5}}\right)$$

$$= 4 \times \frac{3}{5}x^{\frac{3}{5}-1}$$

$$\frac{dy}{dx} = \frac{12}{5}\ x^{\frac{-2}{5}}$$

Answer: ii

Given $y = \frac{1}{\sqrt{x^3}}$

$$= \frac{1}{x^{\frac{3}{2}}} \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 2;\ m = 3\right]$$

$$\boldsymbol{y = x^{-\frac{3}{2}}} \qquad \left[\because \quad A^{-1} = \frac{1}{A}\right]$$

Differentiate, we get

$$\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}-1}$$

$$= -\frac{3}{2}\ x^{\frac{-3-2}{2}}$$

$$= -\frac{3}{2}\ x^{\frac{-5}{2}}$$

$$= -\frac{3}{2}\ \sqrt{x^{-5}} \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 2;\ m = -5\right]$$

$$\frac{dy}{dx} = -\frac{3}{2}\ \times \frac{1}{\sqrt{x^5}} \qquad \left[\because \quad A^{-1} = \frac{1}{A}\right]$$

Answer: iii

Given $y = x^2 \sqrt[3]{x^5}$

$= x^2 \times x^{\frac{5}{3}}$ $\left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 3;\ m = 5\right]$

$= x^{2+\frac{5}{3}}$ $[\because \quad A^m \times A^n = A^{m+n}]$

$= x^{\frac{6+5}{3}}$

$\boldsymbol{y = x^{\frac{11}{3}}}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{11}{3}\ x^{\frac{11}{3}-1}$$

$$= \frac{11}{3}\ x^{\frac{11-3}{3}}$$

$$= \frac{11}{3}\ x^{\frac{8}{3}}$$

$$\frac{dy}{dx} = \frac{11}{3}\ \sqrt[3]{x^8} \qquad \left[\because \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 3;\ m = 8\right]$$

Answer: iv

Given $y = \dfrac{x-3}{\sqrt{x}}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{\sqrt{x} \times \frac{d}{dx}(x-3) - (x-3) \times \frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2} \qquad \left[\because\ d\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}\right]$$

$$= \frac{\sqrt{x}\,(1-0) - (x-3)\left[\frac{1}{2\sqrt{x}}\right]}{x}$$

$$= \frac{\sqrt{x}\ - \frac{(x-3)}{2\sqrt{x}}}{x}$$

$$= \frac{\left(\frac{2\sqrt{x} \times \sqrt{x}\ - (x-3)}{2\sqrt{x}}\right)}{x}$$

$$= \frac{2x - x + 3}{2x\sqrt{x}}$$

$$= \frac{x+3}{2x\sqrt{x}}$$

Answer: v

Given $y = \left(x + \frac{1}{x}\right)^2$

Differentiate, we get

$$\frac{dy}{dx} = 2\left(x + \frac{1}{x}\right)^{2-1} \times \frac{d}{dx}\left(x + \frac{1}{x}\right)$$

$$= 2\left(x + \frac{1}{x}\right)\left(1 + \left[-\frac{1}{x^2}\right]\right)$$

$$= 2\left(\frac{x^2+1}{x}\right)\left(\frac{x^2-1}{x^2}\right) \qquad [\because \quad (A+B)(A-B) = A^2 - B^2]$$

$$= 2\left[\frac{(x^2)^2 - 1^2}{x^3}\right]$$

$$= \frac{2}{x^3}(x^4 - 1)$$

Answer: vi

Given $y = \frac{(2x+3)(x+3)}{x^2}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{x^2 \times \frac{d}{dx}[(2x+3)(x+3)] - [(2x+3)(x+3)] \times \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left[(2x+3) \times \frac{d}{dx}(x+3) + (x+3) \times \frac{d}{dx}(2x+3)\right] - [(2x+3)(x+3)]2x}{x^4}$$

$$= \frac{x^2[(2x+3)(1+0) + (x+3)(2+0)] - [(2x+3)(x+3)]2x}{x^4}$$

$$= x\left[\frac{x(2x+3+2x+6) - 2(2x^2+3x+6x+9)}{x^4}\right]$$

$$= \frac{2x^2 + 3x + 2x^2 + 6x - 4x^2 - 6x - 12x - 18}{x^3}$$

$$= \frac{-9x - 18}{x^3}$$

$$= \frac{-9}{x^3}(x+2)$$

Problem: 2

Differentiate the following $f(x) = \sqrt[5]{7x^3 - 2x^2 + 5}$

Answer:

Given $y = \sqrt[5]{7x^3 - 2x^2 + 5}$

$$y = (7x^3 - 2x^2 + 5)^{\frac{1}{5}} \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 5;\ m = 1\right]$$

Differentiate, we get

$$\frac{dy}{dx} = \frac{1}{5}\,(7x^3 - 2x^2 + 5)^{\frac{1}{5}-1} \times \frac{d}{dx}(7x^3 - 2x^2 + 5)$$

$$= \frac{1}{5}\,(7x^3 - 2x^2 + 5)^{\frac{1-5}{5}} \times [21\,x^2 - 4x + 0]$$

$$= \frac{1}{5}\,(7x^3 - 2x^2 + 5)^{\frac{-4}{5}} \times x[21x - 4]$$

$$= \frac{x}{5}(21x - 4)\,(7x^3 - 2x^2 + 5)^{\frac{-4}{5}}$$

$$= \frac{x(21x - 4)}{5}\,\sqrt[5]{(7x^3 - 2x^2 + 5)^{-4}} \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 5;\ m = -4\right]$$

$$= \frac{x(21x - 4)}{5\sqrt[5]{(7x^3 - 2x^2 + 5)^4}} \qquad \left[\because \ \frac{1}{A} = A^{-1}\right]$$

Problem: 3

Differentiate the following $f(x) = \sqrt{x + \sqrt{x}}$

Answer:

Given $y = \sqrt{x + \sqrt{x}}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x + \sqrt{x}}} \times \frac{d}{dx}\left(x + \sqrt{x}\right)$$

$$= \frac{1}{2\sqrt{x + \sqrt{x}}}\left[\frac{d}{dx}(x) + \frac{d}{dx}\left(\sqrt{x}\right)\right]$$

$$= \frac{1}{2\sqrt{x + \sqrt{x}}}\left[1 + \frac{1}{2\sqrt{x}} \times \frac{d}{dx}(1)\right]$$

$$= \frac{1}{2\sqrt{x+\sqrt{x}}}\left[\frac{2\sqrt{x}+1}{2\sqrt{x}}\right]$$

$$= \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$$

Problem: 4

Differentiate the following $f(x) = (2x+9)\sqrt{x^2-4}$.

Answer:

Given $y = (2x+9)\sqrt{x^2-4}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[(2x+9)\sqrt{x^2-4}\right]$$

$$= (2x+9) \times \frac{d}{dx}\left(\sqrt{x^2-4}\right) + \sqrt{x^2-4} \times \frac{d}{dx}(2x+9)$$

$$= (2x+9)\ \frac{1}{2\sqrt{x^2-4}} \times \frac{d}{dx}(x^2-4) + \sqrt{x^2-4}\,(2+0)$$

$$= (2x+9)\ \frac{1}{2\sqrt{x^2-4}}\ (2x-0) + 2\sqrt{x^2-4}$$

$$= \frac{2x(2x+9)}{2\sqrt{x^2-4}} + 2\sqrt{x^2-4}$$

$$= \frac{x(2x+9)}{\sqrt{x^2-4}} + 2\sqrt{x^2-4}$$

$$= \frac{x(2x+9) + 2\sqrt{x^2-4} \times \sqrt{x^2-4}}{\sqrt{x^2-4}}$$

$$= \frac{2x^2+9x+2(x^2-4)}{\sqrt{x^2-4}}$$

$$= \frac{2x^2+9x+2x^2-8}{\sqrt{x^2-4}}$$

$$= \frac{4x^2+9x-8}{\sqrt{x^2-4}}$$

Problem: 5

Differentiate the following $f(x) = \sqrt[3]{(4x-1)^4}$

Answer:

Given $y = \sqrt[3]{(4x-1)^4}$

$$\boldsymbol{y = (4x-1)^{\frac{4}{3}}} \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 3;\ m = 4\right]$$

Differentiate, we get

$$\frac{dy}{dx} = \frac{4}{3}(4x-1)^{\frac{4}{3}-1} \times \frac{d}{dx}(4x-1)$$

$$= \frac{4}{3}(4x-1)^{\frac{4-3}{3}} \times (4-0)$$

$$= \frac{4}{3}(4x-1)^{\frac{1}{3}} \times 4 \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 3;\ m = 1\right]$$

$$= \frac{16}{3}\sqrt[3]{4x-1}$$

Problem: 6

Differentiate the following $f(x) = (x-3)\sqrt{x-3}$

Answer:

Given $y = (x-3)\sqrt{x-3}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x-3)\sqrt{x-3}\right]$$

$$= (x-3) \times \frac{d}{dx}\left(\sqrt{x-3}\right) + \sqrt{x-3} \times \frac{d}{dx}(x-3)$$

$$= (x-3)\frac{1}{2\sqrt{x-3}} \times \frac{d}{dx}(x-3) + \sqrt{x-3}\,(1-0)$$

$$= \frac{(x-3)}{2\sqrt{x-3}} \times (1-0) + \sqrt{x-3}$$

$$= \frac{\sqrt{x-3} \times \sqrt{x-3}}{2\sqrt{x-3}} + \sqrt{x-3} \qquad \left[\because \quad \sqrt{A} \times \sqrt{A} = A\right]$$

$$= \frac{\sqrt{x-3}}{2} + \sqrt{x-3}$$

$$= \sqrt{x-3}\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{2}\sqrt{x-3}$$

Problem: 7

Differentiate the following $f(x) = \left(7x + \sqrt{x^2+3}\right)^6$

Answer:

Given $y = \left(7x + \sqrt{x^2+3}\right)^6$

Differentiate, we get

$$\frac{dy}{dx} = 6\left(7x + \sqrt{x^2+3}\right)^{6-1} \times \frac{d}{dx}\left(7x + \sqrt{x^2+3}\right)$$

$$= 6\left(7x + \sqrt{x^2+3}\right)^5 \times \left[\frac{d}{dx}(7x) + \frac{d}{dx}\sqrt{x^2+3}\right]$$

$$= 6\left(7x + \sqrt{x^2+3}\right)^5 \times \left[7 + \frac{1}{2\sqrt{x^2+3}} \times \frac{d}{dx}(x^2+3)\right]$$

$$= 6\left(7x + \sqrt{x^2+3}\right)^5 \times \left[7 + \frac{1}{2\sqrt{x^2+3}} \times (2x+0)\right]$$

$$= 6\left(7x + \sqrt{x^2+3}\right)^5 \times \left[7 + \frac{x}{\sqrt{x^2+3}}\right]$$

Problem: 8

Differentiate the following $f(x) = (3x+8)^5(x+1)^4$

Answer:

Given $y = (3x+8)^5(x+1)^4$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(3x+8)^5(x+1)^4]$$

$$= (3x+8)^5 \times \frac{d}{dx}(x+1)^4 + (x+1)^4 \times \frac{d}{dx}(3x+8)^5$$

$$= (3x+8)^5\left[4(x+1)^3 \times \frac{d}{dx}(x+1)\right] + (x+1)^4\left[5(3x+8)^4 \times \frac{d}{dx}(3x+8)\right]$$

$$= (3x+8)^5[4(x+1)^3 \times (1+0)] + (x+1)^4[5(3x+8)^4 \times (3+0)]$$

$$= (3x+8)^5[4(x+1)^3] + (x+1)^4[15(3x+8)^4]$$

$$= (3x+8)^4(x+1)^3[4(3x+8) + 15(x+1)]$$

$$= (3x+8)^4(x+1)^3[12x+32+15x+15]$$

$$= (3x+8)^4(x+1)^3[27x+47]$$

Problem: 9

Differentiate the following $f(x) = (x+1)(x+2)(x+3)$

Answer:

Given $y = (x+1)(x+2)(x+3)$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(x+1)(x+2)(x+3)]$$

$$= (x+1)(x+2)\frac{d}{dx}(x+3) + (x+1)(x+3)\frac{d}{dx}(x+2) + (x+2)(x+3)\frac{d}{dx}(x+1)$$

$$= (x+1)(x+2)(1+0) + (x+1)(x+3)(1+0) + (x+2)(x+3)(1+0)$$

$$= (x+1)(x+2) + (x+1)(x+3) + (x+2)(x+3)$$

Problem: 10

Differentiate the following $f(x) = \left(x^2 + \frac{1}{x^2}\right)^3$

Answer:

Given $y = \left(x^2 + \frac{1}{x^2}\right)^3$

Differentiate, we get

$$\frac{dy}{dx} = 3\left(x^2 + \frac{1}{x^2}\right)^{3-1} \times \frac{d}{dx}\left(x^2 + \frac{1}{x^2}\right)$$

$$= 3\left(x^2 + \frac{1}{x^2}\right)^2 \times \frac{d}{dx}(x^2 + x^{-2}) \qquad \left[\because \quad A^{-1} = \frac{1}{A}\right]$$

$$= 3\left(x^2 + \frac{1}{x^2}\right)^2 \times [2x + (-2x^{-3})]$$

$$= 3\left(x^2 + \frac{1}{x^2}\right)^2 \times 2\left[x - \frac{1}{x^3}\right] \qquad \left[\because \quad A^{-1} = \frac{1}{A}\right]$$

$$= 6\left(x^2 + \frac{1}{x^2}\right)^2 \left[\frac{x^3 - 1}{x^3}\right]$$

Problems based on Exponential functions

Problem: 11

Differentiate the following $f(x) = e^{x^2-2x+7}$.

Answer:

Given $y = e^{x^2-2x+7}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^2-2x+7}\right)$$

$$= e^{x^2-2x+7} \times \frac{d}{dx}(x^2 - 2x + 7) \qquad \left[\because \quad A^{-1} = \frac{1}{A}\right]$$

$$= e^{x^2-2x+7} \times [2x - 2]$$

$$= 2(x-1)e^{x^2-2x+7}$$

Problem: 12

Differentiate the following $f(x) = (2x+1)e^{-x}$.

Answer:

Given $y = (2x+1)e^{-x}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(2x+1)e^{-x}]$$

$$= (2x+1) \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(2x+1) \qquad \left[\because \quad d(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= (2x+1)[e^{-x}(-1)] + e^{-x}(2+0) \qquad [\because \quad d(e^{ax}) = ae^{ax}]$$

$$= e^{-x}[-(2x+1) + 2]$$

$$= e^{-x}[-2x - 1 + 2]$$

$$= e^{-x}[1 - 2x]$$

Problem: 13

Differentiate the following $y = \sqrt{x}\, e^{-x}$.

Answer:

Given $y = \sqrt{x}\, e^{-x}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sqrt{x}\, e^{-x}\right]$$

$$= \sqrt{x} \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\sqrt{x}) \qquad \left[\because \quad d(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= \sqrt{x}\,[e^{-x}(-1)] + e^{-x}\left[\frac{1}{2\sqrt{x}}\right]$$

$$= e^{-x}\left[-\sqrt{x} + \frac{1}{2\sqrt{x}}\right]$$

$$= e^{-x}\left[\frac{-2\sqrt{x} \times \sqrt{x} + 1}{2\sqrt{x}}\right]$$

$$= e^{-x}\left[\frac{-2x + 1}{2\sqrt{x}}\right]$$

Problem: 14

Differentiate the following $f(x) = \frac{e^x + 1}{e^x - 1}$.

Answer:

Given $y = \frac{e^x + 1}{e^x - 1}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + 1}{e^x - 1}\right)$$

$$= \frac{(e^x - 1) \times \frac{d}{dx}(e^x + 1) - (e^x + 1) \times \frac{d}{dx}(e^x - 1)}{(e^x - 1)^2} \qquad \left[\because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{(e^x - 1)(e^x + 0) - (e^x + 1)(e^x - 0)}{(e^x - 1)^2}$$

$$= \frac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2}$$

$$= \frac{e^x[e^x - 1 - e^x - 1]}{(e^x - 1)^2}$$

$$= \frac{-2e^x}{(e^x - 1)^2}$$

Problem: 15

Differentiate the following $f(x) = e^{\sqrt[4]{x}} - e^{-\frac{1}{x}}$.

Answer:

Given $y = e^{\sqrt[4]{x}} - e^{-\frac{1}{x}}$

$$y = e^{x^{\frac{1}{4}}} - e^{-\frac{1}{x}} \qquad \left[\because \quad \sqrt[a]{x^m} = x^{\frac{m}{a}}; \quad a = 4;\ m = 1\right]$$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^{\frac{1}{4}}} - e^{-\frac{1}{x}}\right)$$

$$= e^{x^{\frac{1}{4}}} \times \frac{d}{dx}\left(x^{\frac{1}{4}}\right) - e^{-\frac{1}{x}} \times \frac{d}{dx}\left(-\frac{1}{x}\right)$$

$$= e^{x^{\frac{1}{4}}}\left[\frac{1}{4}\, x^{\frac{1}{4}-1}\right] - e^{-\frac{1}{x}}\left[-\left(\frac{-1}{x^2}\right)\right]$$

$$= \frac{1}{4} e^{x^{\frac{1}{4}}}\, x^{\frac{-3}{4}} - \frac{1}{x^2} e^{-\frac{1}{x}}$$

Problem: 16

Differentiate the following $f(x) = (e^x + 2)^8$.

Answer:

Given $y = (e^x + 2)^8$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^x + 2)^8$$

$$= 8\,(e^x + 2)^{8-1} \times \frac{d}{dx}(e^x + 2)$$

$$= 8\,(e^x + 2)^7 \times (e^x + 0)$$

$$= 8\, e^x (e^x + 2)^7$$

Problems contains Trigonometric functions

Problem: 17

Differentiate the following $y = \tan(x^2 - 1)$.

Answer:

Given $y = \tan(x^2 - 1)$

Differentiate, we get

$$\frac{dy}{dx} = \sec^2(x^2 - 1) \times \frac{d}{dx}(x^2 - 1)$$

$$= \sec^2(x^2 - 1) \times (2x - 0)$$

$$= 2x \sec^2(x^2 - 1)$$

Problem: 18

Differentiate the following $y = \tan\left(\sqrt{\cos x}\right)$.

Answer:

Given $y = \tan\left(\sqrt{\cos x}\right)$

Differentiate, we get

$$\frac{dy}{dx} = \sec^2\left(\sqrt{\cos x}\right) \times \frac{d}{dx}\left(\sqrt{\cos x}\right)$$

$$= \sec^2\left(\sqrt{\cos x}\right) \times \frac{1}{2\sqrt{\cos x}} \times \frac{d}{dx}(\cos x)$$

$$= \sec^2\left(\sqrt{\cos x}\right) \times \frac{1}{2\sqrt{\cos x}} \times (-\sin x)$$

$$= \frac{-\sin x \ \sec^2\left(\sqrt{\cos x}\right)}{2\sqrt{\cos x}}$$

Problem: 19

Differentiate the following $y = \sin^4 \sqrt{x}$.

Answer:

Given $y = \sin^4 \sqrt{x}$

$$= \left(\sin \sqrt{x}\right)^4$$

Differentiate, we get

$$\frac{dy}{dx} = 4\left(\sin\sqrt{x}\right)^{4-1} \times \frac{d}{dx}\left(\sin\sqrt{x}\right)$$

$$= 4\left(\sin\sqrt{x}\right)^{3} \times \cos\sqrt{x} \times \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= 4\sin^3\sqrt{x} \times \cos\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{2}{\sqrt{x}}\sin^3\sqrt{x}\cos\sqrt{x}$$

Problem: 20

Differentiate the following $y = \sqrt[3]{\frac{\tan x}{x}}$.

Answer:

Given $y = \sqrt[3]{\frac{\tan x}{x}}$ $\left[\because \sqrt[a]{x^m} = x^{\frac{m}{a}};\quad a = 3;\ m = 1\right]$

$$= \left(\frac{\tan x}{x}\right)^{\frac{1}{3}}$$

Differentiate, we get

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{\tan x}{x}\right)^{\frac{1}{3}-1} \times \frac{d}{dx}\left(\frac{\tan x}{x}\right)$$

$$= \frac{1}{3}\left(\frac{\tan x}{x}\right)^{\frac{1-3}{3}} \times \left[\frac{x \times \frac{d}{dx}(\tan x) - \tan x \times \frac{d}{dx}(x)}{x^2}\right]$$

$$= \frac{1}{3}\left(\frac{\tan x}{x}\right)^{\frac{-2}{3}} \times \left[\frac{x\sec^2 x - \tan x\,(1)}{x^2}\right] \qquad \left[\because \sqrt[a]{x^m} = x^{\frac{m}{a}};\quad a = 3;\ m = -2\right]$$

$$= \frac{1}{3}\sqrt[3]{\left(\frac{\tan x}{x}\right)^{-2}} \times \left[\frac{x\sec^2 x - \tan x}{x^2}\right]$$

$$= \frac{1}{3}\sqrt[3]{\left(\frac{x}{\tan x}\right)^{2}} \times \left[\frac{x\sec^2 x - \tan x}{x^2}\right] \qquad \left[\because \frac{1}{A} = A^{-1}\right]$$

Problem: 21

Differentiate the following $y = \frac{\sin x}{x^2}$.

Answer:

Given $y = \frac{\sin x}{x^2}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x^2}\right)$$

$$= \frac{x^2 \times \frac{d}{dx}(\sin x) - \sin x \times \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(\cos x) - \sin x\ (2x)}{x^4}$$

$$= x\left[\frac{x\cos x - 2\sin x}{x^4}\right]$$

$$= \frac{x\cos x - 2\sin x}{x^3}$$

Problem: 22

Differentiate the following $y = x^2 \tan 8x$.

Answer:

Given $y = x^2 \tan 8x$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \tan 8x)$$

$$= x^2 \frac{d}{dx}(\tan 8x) - \tan 8x\ \frac{d}{dx}(x^2)$$

$$= x^2\left[\sec^2 8x \times \frac{d}{dx}(8x)\right] - \tan 8x\ (2x)$$

$$= x^2[\sec^2 8x \times 8] - 2x\ \tan 8x$$

$$= 2x[4x\sec^2 8x - \tan 8x]$$

Problem: 23

Differentiate the following $y = \sin 2x \tan 3x$.

Answer:

Given $y = \sin 2x \tan 3x$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 2x \tan 3x)$$

$$= \sin 2x \times \frac{d}{dx}(\tan 3x) - \tan 3x \times \frac{d}{dx}(\sin 2x)$$

$$= \sin 2x \left[\sec^2 3x \times \frac{d}{dx}(3x)\right] - \tan 3x \left[\cos 2x \times \frac{d}{dx}(2x)\right]$$

$$= \sin 2x [\sec^2 3x \times 3] - \tan 3x \; [\cos 2x \times 2]$$

$$= 3 \sin 2x \sec^2 3x - 2 \tan 3x \; \cos 2x$$

Problems based on Logarithmic functions

Problem: 24

Differentiate the following $y = \ln(x^2 - 5)$.

Answer:

Given $y = \ln(x^2 - 5)$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(x^2 - 5)]$$

$$= \frac{1}{x^2 - 5} \times \frac{d}{dx}(x^2 - 5)$$

$$= \frac{1}{x^2 - 5} \times (2x - 0)$$

$$= \frac{2x}{x^2 - 5}$$

Problem: 25

Differentiate the following $y = \ln \sqrt{x + 9}$.

Answer:

Given $y = \ln \sqrt{x + 9}$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\sqrt{x+9}\right]$$

$$= \frac{1}{\sqrt{x+9}} \times \frac{d}{dx}\left(\sqrt{x+9}\right)$$

$$= \frac{1}{\sqrt{x+9}} \times \frac{1}{2\sqrt{x+9}} \times \frac{d}{dx}(x+9) \qquad \left[\because \ \sqrt{A}\times\sqrt{A} = A\right]$$

$$= \frac{1}{2(x+9)}(1+0)$$

$$= \frac{1}{2(x+9)}$$

Problem: 26

Differentiate the following $y = \ln(x+6)^4$.

Answer:

Given $y = \ln(x+6)^4$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\ln[(x+6)^4]\}$$

$$= \frac{1}{(x+6)^4} \times \frac{d}{dx}[(x+6)^4]$$

$$= \frac{1}{(x+6)^4} \times 4(x+6)^3 \times \frac{d}{dx}(x+6)$$

$$= \frac{4}{(x+6)} \times (1+0)$$

$$= \frac{4}{x+6}$$

Problem: 27

Differentiate the following $y = \ln\left(\frac{x+1}{x+3}\right)$.

Answer:

Given $y = \ln\left(\frac{x+1}{x+3}\right)$

$$y = \ln(x+1) - \ln(x+3) \qquad \left[\because \ \log\left(\frac{A}{B}\right) = \log A - \log B\right]$$

Differentiate, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(x+1) - \ln(x+3)]$$

$$= \frac{1}{x+1} \times \frac{d}{dx}(x+1) - \frac{1}{x+3} \times \frac{d}{dx}(x+3)$$

$$= \frac{1}{x+1} \times (1+0) - \frac{1}{x+3} \times (1+0)$$

$$= \frac{1}{x+1} - \frac{1}{x+3}$$

Problems based on Implicit differentiation method

Problem: 28

Find the first derivative of the circle with centre (0,0) and radius r.

Answer:

The equation of circle with centre (0,0) and radius r is

$$x^2 + y^2 = r^2$$

Differentiate both side with respect to 'x' we get

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(r^2)$$

$$2x + 2y \times \frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\boldsymbol{\frac{dy}{dx} = -\frac{x}{y}}$$

Problem: 29

Find the first derivative of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer:

The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiate both side with respect to $'x'$ we get

$$\frac{1}{a^2}\frac{d}{dx}(x^2)+\frac{1}{b^2}\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$$

$$\frac{1}{a^2}(2x)+\frac{1}{b^2}\left(2y\times\frac{dy}{dx}\right)=0$$

$$\frac{2y}{b^2}\frac{dy}{dx}=-\frac{1}{a^2}(2x)$$

$$\frac{dy}{dx}=-\frac{2x}{a^2}\times\frac{b^2}{2y}$$

$$\boldsymbol{\frac{dy}{dx}=-\frac{b^2}{a^2}\times\frac{x}{y}}$$

Problem: 30

Find the first derivative of the parabola $y^2=4ax$.

Answer:

The equation of the parabola is

$$y^2=4ax$$

Differentiate both side with respect to $'x'$ we get

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(4ax)$$

$$2y\times\frac{dy}{dx}=4a(1)$$

$$\frac{dy}{dx}=\frac{4a}{2y}$$

$$\boldsymbol{\frac{dy}{dx}=\frac{2a}{y}}$$

Problem: 31

Find the first derivative of the equation $x^3+y^3=3xy$.

Answer:

Given the equation

$$x^3+y^3=3xy$$

Differentiate both side with respect to $'x'$ we get

$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)=\frac{d}{dx}(3xy)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3\left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right]$$

$$3x^2 + 3y^2\,\frac{dy}{dx} = 3\left[x\frac{dy}{dx} + y\right]$$

$$3x^2 + 3y^2\,\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$$

$$\frac{dy}{dx}[3y^2 - 3x] = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}}$$

Problem: 32

Find the first derivative of the equation $8x^3e^{y^2} = 3$ by implicit differentiation.

Answer:

Consider the equation

$$8x^3e^{y^2} = 3 \qquad \ldots(1)$$

Differentiate both side with respect to $'x'$ we get

$$8\,\frac{d}{dx}\left(x^3e^{y^2}\right) = \frac{d}{dx}(3) \qquad \left[\because \quad d(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right]$$

By using differentiation of product rule, we get

$$8\left[x^3 \times \frac{d}{dx}\left(e^{y^2}\right) + e^{y^2} \times \frac{d}{dx}(x^3)\right] = \frac{d}{dx}(3)$$

$$8\left[x^3\left(e^{y^2} \times \frac{d}{dx}(y^2)\right) + e^{y^2} \times 3x^2\right] = 0$$

$$x^3\left(e^{y^2} \times 2y \times \frac{dy}{dx}\right) + e^{y^2} \times 3x^2 = \frac{0}{8}$$

$$e^{y^2}\left[2y\,x^3\frac{dy}{dx} + 3x^2\right] = 0$$

$$2y\,x^3\frac{dy}{dx} + 3x^2 = \frac{0}{e^{y^2}}$$

$$2y\,x^3\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = -\frac{3x^2}{2y\,x^3}$$

$$\boldsymbol{\frac{dy}{dx} = -\frac{3}{2xy}}$$

Problem: 33

Find the first derivative of the equation $\frac{y}{x} = \ln(xy)$.

Answer:

Given the equation

$$\frac{y}{x} = \ln(xy) \qquad [\because \log(A \times B) = \log A + \log B]$$

$$\frac{y}{x} = \ln x + \ln y \qquad \ldots (1)$$

Differentiate both side with respect to $'x'$ we get

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}(\ln x) + \frac{d}{dx}(\ln y)$$

By differentiation of product rule, we get

$$\frac{x \times \frac{d}{dx}(y) - y \times \frac{d}{dx}(x)}{x^2} = \frac{1}{x} + \frac{1}{y} \times \frac{dy}{dx}$$

$$\frac{1}{x^2}\left[x\frac{dy}{dx} - y\right] = \frac{1}{x} + \frac{1}{y} \times \frac{dy}{dx}$$

$$x\frac{dy}{dx} - y = x^2\left[\frac{1}{x} + \frac{1}{y} \times \frac{dy}{dx}\right]$$

$$x\frac{dy}{dx} - y = x + \frac{x^2}{y}\frac{dy}{dx}$$

$$-\frac{x^2}{y}\frac{dy}{dx} + x\frac{dy}{dx} = y + x$$

$$\frac{dy}{dx}\left[-\frac{x^2}{y} + x\right] = y + x$$

$$\frac{dy}{dx} = \frac{y + x}{\left[\frac{-x^2 + xy}{y}\right]}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{y(y + x)}{x(-x + y)}}$$

Problem: 34

Differentiate the equation $a^x + a^y = a^{x+y}$ by implicit differentiation.

Answer:

Consider the equation

$$a^x + a^y = a^{x+y} \qquad [\because a^{m+n} = a^m \times a^n]$$

$$a^x + a^y = a^x \times a^y \qquad \dots(1)$$

Differentiate both side with respect to $'x'$ we get

$$\frac{d}{dx}(a^x) + \frac{d}{dx}(a^y) = \frac{d}{dx}(a^x \times a^y) \qquad \left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$a^x \ln a + a^y \ln a \times \frac{dy}{dx} = a^x \frac{d}{dx}(a^y) + a^y \frac{d}{dx}(a^x)$$

$$a^x \ln a + a^y \ln a \times \frac{dy}{dx} = a^x \times a^y \ln a \times \frac{dy}{dx} + a^y \times a^x \ln a$$

Combining the terms, we have

$$a^y \ln a \times \frac{dy}{dx} - a^x a^y \ln a \times \frac{dy}{dx} = -a^x \ln a + a^y a^x \ln a$$

$$a^y \ln a \frac{dy}{dx}[1 - a^x] = -a^x \ln a [1 - a^y]$$

$$\frac{dy}{dx} = \frac{-a^x \ln a [1 - a^y]}{a^y \ln a [1 - a^x]}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{-a^x[1 - a^y]}{a^y[1 - a^x]}}$$

Problem: 35

Find the first derivative of the equation $y^4 + xy = 10$ by implicit differentiation.

Answer:

Consider the equation

$$y^4 + xy = 10$$

Differentiate both side with respect to $'x'$ we get

$$\frac{d}{dx}(y^4) + \frac{d}{dx}(xy) = \frac{d}{dx}(10) \qquad \left[\because \quad d(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right]$$

$$4y^3 \times \frac{dy}{dx} + \left[x \times \frac{d}{dx}(y) + y \times \frac{d}{dx}(x)\right] = 0$$

$$4y^3\frac{dy}{dx}+x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}[4y^3+x]=-y$$

$$\boldsymbol{\frac{dy}{dx}=\frac{-y}{4y^3+x}}$$

Logarithmic Differentiation

Logarithmic differentiation is a method to find the derivatives of some complicated functions, using logarithms.

The method of differentiating functions by first taking logarithms and then differentiating is called logarithmic differentiation.

Problem: 1

Find the first derivative of the equation $x^y=y^x$.

Answer:

Given the equation

$$x^y=y^x \qquad \dots(1)$$

The direct differentiation of above equation is difficult.

In this case we use logarithmic differentiation method.

Taking log on both sides, we get

$$\ln x^y=\ln y^x \qquad [\because \quad \log A^B=B\log A]$$

$$y\ln x=x\ln y \qquad \dots(2)$$

Differentiate both side with respect to $'x'$ we get

$$\frac{d}{dx}(y\ln x)=\frac{d}{dx}(x\ln y)$$

By differentiation of product rule, we get

$$y\times\frac{d}{dx}(\ln x)+\ln x\times\frac{d}{dx}(y)=x\times\frac{d}{dx}(\ln y)+\ln y\times\frac{d}{dx}(x)$$

$$y\left(\frac{1}{x}\right)+\ln x\frac{dy}{dx}=x\left(\frac{1}{y}\right)\times\frac{dy}{dx}+\ln y\,(1)$$

$$y\left(\frac{1}{x}\right) + \ln x \frac{dy}{dx} - x\left(\frac{1}{y}\right) \times \frac{dy}{dx} = + \ln y\,(1)$$

$$\frac{dy}{dx}\left[\ln x - \frac{x}{y}\right] = -\frac{y}{x} + \ln y$$

$$\frac{dy}{dx} = \frac{-\frac{y}{x} + \ln y}{\ln x - \frac{x}{y}}$$

$$= \frac{\left(\frac{-y + x\ln y}{x}\right)}{\left(\frac{y\ln x - x}{y}\right)}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{y(-y + x\ln y)}{x(y\ln x - x)}}$$

Problem: 2

Find the derivative of $y = x^{\sin x}$

Answer:

$y = x^{\sin x}$ ….(1)

The direct differentiation of above equation is difficult.

In this case we use logarithmic differentiation method.

Taking log on both sides, we get

$\ln y = \ln\left(x^{\sin x}\right)$ $[\because \log A^B = B\log A]$

$\ln y = \sin x \ln x$ ….(2)

Now we can differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\sin x \ln x] \qquad [\because d(uv) = u\,v' + v\,u']$$

$$\frac{1}{y} \times \frac{dy}{dx} = \sin x \times \frac{d}{dx}[\ln x] + \ln x \times \frac{d}{dx}[\sin x]$$

$$\frac{1}{y} \times \frac{dy}{dx} = \sin x\left(\frac{1}{x}\right) + \ln x \cos x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x\ \ln x\right]$$

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x\ \ln x\right] \qquad [\because\ y = x^{\sin x}]$$

Problem: 3

Find the derivative of $y = (\sin x)^x$

Answer:

$$y = (\sin x)^x \qquad \text{....}(1)$$

Taking log on both sides, we get

$$\ln y = \ln(\sin x)^x \qquad [\because \log A^B = B \log A]$$

$$\ln y = x \ln(\sin x) \qquad \text{....}(2)$$

Now we can differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[x \ln(\sin x)] \qquad [\because d(uv) = u\,v' + v\,u']$$

$$\frac{1}{y} \times \frac{dy}{dx} = x \times \frac{d}{dx}(\ln \sin x) + \ln(\sin x) \times \frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x) + \ln(\sin x)$$

$$\frac{dy}{dx} = y\left[x \frac{1}{\sin x} \cos x + \ln(\sin x)\right]$$

$$\frac{dy}{dx} = y\left[x \frac{\cos x}{\sin x} + \ln(\sin x)\right] \qquad [\because y = (\sin x)^x]$$

$$\frac{dy}{dx} = (\sin x)^x [x \cot x + \ln(\sin x)]$$

Problem: 4

Find the derivative of $y = x^{\sqrt{x}}$.

Answer:

$$\boldsymbol{y = x^{\sqrt{x}}} \qquad \text{....}(\mathbf{1})$$

Taking log on both sides, we get

$$\ln y = \ln x^{\sqrt{x}} \qquad [\because \log A^B = B \log A]$$

$$\mathbf{\ln y = \sqrt{x}\ \ln x} \qquad \text{....}(\mathbf{2})$$

Now differentiate (2) by using any of the differentiation rule.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\sqrt{x}\ \ln x] \qquad [\because d(uv) = u\,v' + v\,u']$$

$$\frac{1}{y} \times \frac{dy}{dx} = \sqrt{x} \times \frac{d}{dx}(\ln x) + \ln x \times \frac{d}{dx}(\sqrt{x})$$

$$\frac{dy}{dx} = y\left[\sqrt{x} \times \frac{1}{x} + \ln x \times \frac{1}{2\sqrt{x}}\right]$$

$$\frac{dy}{dx} = y\left[\frac{\sqrt{x}}{\sqrt{x} \times \sqrt{x}} + \ln x \frac{1}{2\sqrt{x}}\right] \qquad [\because\ x = \sqrt{x} \times \sqrt{x}]$$

$$\frac{dy}{dx} = y\left[\frac{1}{\sqrt{x}} + \ln x \frac{1}{2\sqrt{x}}\right]$$

$$\boldsymbol{\frac{dy}{dx} = x^{\sqrt{x}}\left[\frac{2 + \ln x}{2\sqrt{x}}\right]} \qquad [\because\ y = x^{\sqrt{x}}]$$

Problem: 5

Find the derivative of $y = x^{\ln x}$.

Answer:

$$\boldsymbol{y = x^{\ln x}} \qquad \dots.(1)$$

Taking log on both sides, we get

$$\ln y = \ln\left(x^{\ln x}\right) \qquad [\because\ \log A^B = B \log A]$$

$$\ln y = \ln x \times \ln x$$

$$\boldsymbol{\ln y = (\ln x)^2} \qquad \dots.(2)$$

Now differentiate (2) by using any of the differentiation rule.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln x)^2$$

$$\frac{1}{y} \times \frac{dy}{dx} = 2\ln x \times \frac{d}{dx}\ln x$$

$$\frac{dy}{dx} = y\left[2\ln x \times \frac{1}{x}\right]$$

$$\boldsymbol{\frac{dy}{dx} = x^{\ln x}\left[\frac{2\ln x}{x}\right]} \qquad [\because\ y = x^{\ln x}]$$

Problem: 6

Find the derivative of $y = (\ln x)^x$.

Answer:

$$\boldsymbol{y = (\ln x)^x} \qquad \dots.(1)$$

Taking log on both sides, we get

$$\ln y = \ln[(\ln x)^x] \qquad [\because\ \log A^B = B \log A]$$

$$\boldsymbol{\ln y = x\ \ln(\ln x)} \qquad \ldots.(\mathbf{2})$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[x \ln(\ln x)]$$

$$\frac{1}{y} \times \frac{dy}{dx} = x \times \frac{d}{dx}[\ln(\ln x)] + \ln(\ln x)\ \times \frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\ln x} \times \frac{d}{dx}[\ln x] + \ln(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{\ln x} \times \frac{1}{x} + \ln(\ln x)$$

$$\frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

$$\boldsymbol{\frac{dy}{dx} = (\ln x)^x \left[\frac{1}{\ln x} + \ln(\ln x)\right]} \qquad [\because\ y = (\ln x)^x]$$

Problem: 7

Find the derivative of $y = x^{\tan x}$.

Answer:

$$\boldsymbol{y = x^{\tan x}} \qquad \ldots.(\mathbf{1})$$

Taking log on both sides, we get

$$\ln y = \ln[x^{\tan x}] \qquad [\because\ \log A^B = B \log A]$$

$$\boldsymbol{\ln y = \tan x\ \ln(x)} \qquad \ldots.(\mathbf{2})$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\tan x \times \ln x]$$

$$\frac{1}{y} \times \frac{dy}{dx} = \tan x \times \frac{d}{dx}[\ln x] + \ln x\ \times \frac{d}{dx}(\tan x)$$

$$\frac{1}{y}\frac{dy}{dx} = \tan x \times \frac{1}{x} + \ln x \times \sec^2 x$$

$$\frac{dy}{dx} = y\left[\frac{\tan x}{x} + \sec^2 x\ \ln x\right]$$

$$\boldsymbol{\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x\ \ln x\right]} \qquad [\because\ y = x^{\tan x}]$$

Problem: 8

Find the derivative of $y = (\cos x)^{\sqrt{x}}$.

Answer:

$$\boldsymbol{y = (\cos x)^{\sqrt{x}}} \qquad \ldots.(1)$$

Taking log on both sides, we get

$$\ln y = \ln\left[(\cos x)^{\sqrt{x}}\right] \qquad [\because \log A^B = B \log A]$$

$$\boldsymbol{\ln y = \sqrt{x} \times \ln(\cos x)} \qquad \ldots.(2)$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\sqrt{x} \times \ln(\cos x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x} \times \frac{d}{dx}[\ln \cos x] + \ln \cos x \times \frac{d}{dx}(\sqrt{x})$$

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x} \times \frac{1}{\cos x} \times \frac{d}{dx}(\cos x) + \ln \cos x \times \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = y\left[\sqrt{x}\,\frac{1}{\cos x}\,(-\sin x) + \frac{\ln \cos x}{2\sqrt{x}}\right]$$

$$\boldsymbol{\frac{dy}{dx} = \cos x^{\sqrt{x}}\left[-\sqrt{x}\tan x + \frac{\ln \cos x}{2\sqrt{x}}\right]} \qquad [\because y = \cos x^{\sqrt{x}}]$$

Problem: 9

Find the derivative of $y = (\sin x)^{\cos x}$.

Answer:

$$\boldsymbol{y = (\sin x)^{\cos x}} \qquad \ldots.(1)$$

Taking log on both sides, we get

$$\ln y = \ln[(\sin x)^{\cos x}] \qquad [\because \log A^B = B \log A]$$

$$\boldsymbol{\ln y = \cos x \times \ln(\sin x)} \qquad \ldots.(2)$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\cos x \times \ln(\sin x)]$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{d}{dx}[\ln \sin x] + \ln \sin x \times \frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x) + \ln \sin x \times (-\sin x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{\sin x}\cos x + \ln \sin x \times (-\sin x)\right]$$

$$\boldsymbol{\frac{dy}{dx} = (\sin x)^{\cos x}[\cot x \, \cos x - \sin x \ln \sin x]} \qquad [\because \; y = (\sin x)^{\cos x}]$$

Problem: 10

Find the derivative of $y = \dfrac{x^x}{(x-1)^2}$.

Answer:

$$\boldsymbol{y = \frac{x^x}{(x-1)^2}} \qquad \dots.(\mathbf{1})$$

Taking log on both sides, we get

$$\ln y = \ln\left[\frac{x^x}{(x-1)^2}\right] \qquad \left[\because \; \log\left(\frac{A}{B}\right) = \log A - \log B\right]$$

$$\ln y = \ln x^x - \ln(x-1)^2 \qquad [\because \; \log A^B = B \log A]$$

$$\boldsymbol{\ln y = x \ln x - 2\ln(x-1)} \qquad \dots.(\mathbf{2})$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[x \times \ln x] - 2\frac{d}{dx}[\ln(x-1)]$$

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{d}{dx}[\ln x] + \ln x \times \frac{d}{dx}(x) - 2\frac{1}{(x-1)} \times \frac{d}{dx}(x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times (1) - 2\frac{1}{(x-1)} \times (1-0)$$

$$\frac{dy}{dx} = y\left[1 + \ln x - \frac{2}{(x-1)}\right]$$

$$\boldsymbol{\frac{dy}{dx} = \frac{x^x}{(x-1)^2}\left[1 + \ln x - \frac{2}{(x-1)}\right]} \qquad \left[\because \; y = \frac{x^x}{(x-1)^2}\right]$$

Problem: 11

Find the derivative of $y = \dfrac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}$.

Answer:

$$\boldsymbol{y = \frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}} \qquad \dots.(1)$$

Taking log on both sides, we get

$$\ln y = \ln\left[\frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}\right] \qquad \left[\because\ \log\left(\frac{A}{B}\right) = \log A - \log B\right]$$

$$\ln y = \ln[(1-x)(2+x)^2(3-x)^3] - \ln(4+x)^4 \qquad [\because\ \log(AB) = \log A + \log B]$$

$$\ln y = \ln(1-x) + \ln(2+x)^2 + \ln(3-x)^3 - \ln(4+x)^4 \qquad [\because\ \log A^B = B\log A]$$

$$\boldsymbol{\ln y = \ln(1-x) + 2\ln(2+x) + 3\ln(3-x) - 4\ln(4+x)} \qquad \dots.(2)$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\ln(1-x)] + 2\frac{d}{dx}[\ln(2+x)] + 3\frac{d}{dx}[\ln(3-x)] - 4\frac{d}{dx}[4+x]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1-x}\frac{d}{dx}(1-x) + \frac{2}{2+x}\frac{d}{dx}(2+x) + \frac{3}{3-x}\frac{d}{dx}(3-x) - \frac{4}{4+x}\frac{d}{dx}[4+x]$$

$$\frac{dy}{dx} = y\left[\frac{1}{1-x}(0-1) + \frac{2}{2+x}(0+1) + \frac{3}{3-x}(0-1) - \frac{4}{4+x}(0+1)\right]$$

$$\boldsymbol{\frac{dy}{dx} = \frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}\left[\frac{-1}{1-x} + \frac{2}{2+x} - \frac{3}{3-x} - \frac{4}{4+x}\right]}$$

$$\left[\because\ y = \frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}\right]$$

Problem: 12

Find the derivative of $y = \dfrac{\sqrt{c+x}\sqrt{c+2x}\sqrt{c+3x}}{\sqrt{c+6x}}$.

Answer:

$$\boldsymbol{y = \frac{\sqrt{c+x}\sqrt{c+2x}\sqrt{c+3x}}{\sqrt{c+6x}}} \qquad \dots.(1)$$

Taking log on both sides, we get

$$\ln y = \ln\left[\frac{\sqrt{c+x}\sqrt{c+2x}\sqrt{c+3x}}{\sqrt{c+6x}}\right] \qquad \left[\because \log\left(\frac{A}{B}\right) = \log A - \log B\right]$$

$$\ln y = \ln\left[\sqrt{c+x}\sqrt{c+2x}\sqrt{c+3x}\right] - \ln\left[\sqrt{c+6x}\right] \qquad [\because \log(AB) = \log A + \log B]$$

$$\ln y = \ln\sqrt{c+x} + \ln\sqrt{c+2x} + \ln\sqrt{c+3x} - \ln\sqrt{c+6x} \qquad \left[\because \sqrt{A} = A^{\frac{1}{2}}\right]$$

$$\ln y = \ln(c+x)^{\frac{1}{2}} + \ln(c+2x)^{\frac{1}{2}} + \ln(c+3x)^{\frac{1}{2}} - \ln(c+6x)^{\frac{1}{2}} \qquad [\because \log A^B = B\log A]$$

$$\boldsymbol{\ln y = \frac{1}{2}\ln(c+x) + \frac{1}{2}\ln(c+2x) + \frac{1}{2}\ln(c+3x) - \frac{1}{2}\ln(c+6x)} \qquad \dots.(2)$$

Now differentiate (2) by using any of the differentiation rules.

$$\frac{d}{dx}(\ln y) = \frac{1}{2}\left[\frac{d}{dx}\ln(c+x) + \frac{d}{dx}\ln(c+2x) + \frac{d}{dx}\ln(c+3x) - \frac{d}{dx}\ln(c+6x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{c+x}(0+1) + \frac{1}{c+2x}(0+2) + \frac{1}{c+3x}(0+3) - \frac{1}{c+6x}(0+6)\right]$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{c+x} + \frac{2}{c+2x} + \frac{3}{c+3x} - \frac{6}{c+6x}\right]$$

$$\boldsymbol{\frac{dy}{dx} = \frac{\sqrt{c+x}\sqrt{c+2x}\sqrt{c+3x}}{2\sqrt{c+6x}}\left[\frac{1}{c+x} + \frac{2}{c+2x} + \frac{3}{c+3x} - \frac{6}{c+6x}\right]}$$

$$\left[\because y = \frac{\sqrt{c+x}\sqrt{c+2x}\sqrt{c+3x}}{\sqrt{c+6x}}\right]$$

Maxima and minima

Monotonicity of functions

Monotonicity of functions are its behavior of increasing or decreasing.

A function $f(x)$ is said to be an increasing function in an interval I

$$\text{if } a < b \iff f(a) \le f(b); \quad \forall\, a, b \in I$$

A function $f(x)$ is said to be decreasing function in an interval I

$$\text{if } a < b \iff f(a) \ge f(b); \quad \forall\, a, b \in I$$

Results:

If the function $f(x)$ is differentiable in an open interval $I = (\mathrm{a,b})$ then

a. the function $f(x)$ is **increasing** in I, iff

$$\frac{d}{dx}[f(x)] \geq 0; \quad \forall\, x \in (a,b)$$

b. the function $f(x)$ is **strictly increasing** in I, iff

$$\frac{d}{dx}[f(x)] > 0; \quad \forall\, x \in (a,b)$$

c. the function $f(x)$ is **decreasing** in I, iff

$$\frac{d}{dx}[f(x)] \leq 0; \quad \forall\, x \in (a,b)$$

d. the function $f(x)$ is **strictly decreasing** in I, iff

$$\frac{d}{dx}[f(x)] < 0; \quad \forall\, x \in (a,b)$$

Critical Points

Let $f(x)$ be a function and let c be a point in the domain of the function. The point c is called a critical point of $f(x)$ if either $f'(c) = 0$ or $f'(c)$ does not exist.

Problem: 1

Determine all the critical points for the function $f(x) = 2x - x^2$.

Answer:

Given $\quad f(x) = 2x - x^2 \quad \ldots(1)$

Differentiate with respect to $'x'$, we get

$$f'(x) = 2 - 2x$$
$$= 2(1 - x)$$

To find the critical point, put $f'(x) = 0$, we get

$$f'(x) = 2(1 - x) = 0$$
$$1 - x = 0$$
$$x = 1$$

The critical point is $\boldsymbol{x = 1}$.

Problem: 2

Determine all the critical points for the function $f(x) = \dfrac{x^2+1}{x^2-x-6}$.

Answer:

Given $\quad f(x) = \dfrac{x^2+1}{x^2-x-6} \qquad \dots (1)$

Differentiate with respect to $'x'$, we get

$$f'(x) = \frac{(x^2-x-6)\times(2x+0)-(x^2+1)\times(2x-1-0)}{(x^2-x-6)^2}$$

$$= \frac{2x^3-2x^2-12x-(2x^3+2x-x^2-1)}{(x^2-x-6)^2}$$

$$f'(x) = \frac{-x^2-14x+1}{(x^2-x-6)^2}$$

To find the critical point, put $f'(x) = 0$, we get

$$f'(x) = \frac{-x^2-14x+1}{(x^2-x-6)^2} = 0$$

$$-x^2-14x+1 = 0$$

Solving, we get

$$x = -7 \pm 5\sqrt{2}$$

The critical points are $\boldsymbol{x = -7+5\sqrt{2}}$, $\boldsymbol{x = -7-5\sqrt{2}}$.

Problem: 3

Determine all the critical points for the function $f(x) = 10\,x\,e^{3-x^2}$.

Answer:

Given $\quad f(x) = 10\,x\,e^{3-x^2} \qquad \dots (1)$

Differentiate with respect to $'x'$, we get

$$f'(x) = 10\left[x \times \frac{d}{dx}\left(e^{3-x^2}\right) + e^{3-x^2} \times \frac{d}{dx}(x)\right]$$

$$= 10\left[x \times e^{3-x^2} \times \frac{d}{dx}(3-x^2) + e^{3-x^2} \times 1\right]$$

$$= 10\left[xe^{3-x^2} \times (0-2x) + e^{3-x^2}\right]$$

$$= 10e^{3-x^2}[-2x^2+1]$$

To find the critical point, put $f'(x) = 0$, we get

$$10e^{3-x^2}[-2x^2 + 1] = 0$$

$$-2x^2 + 1 = 0$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

The critical points are $\boldsymbol{x} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, \quad -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$

Saddle Point

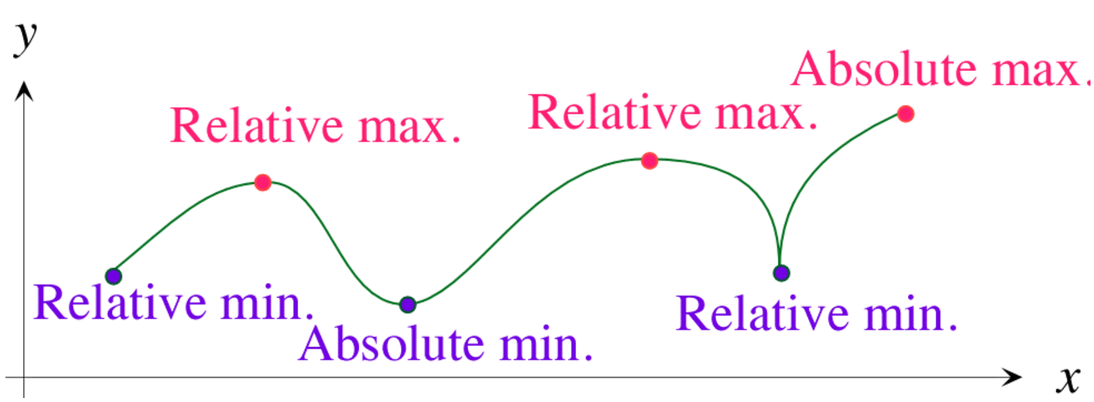
A critical point of a function of a single variable is neither a local maximum, nor a local minimum is called saddle point of the function.

Absolute maxima and minima

The absolute maxima and absolute minima are referred to describing the largest and smallest values of a function on an interval.

Definition

Let x_0 be a number in the domain D of a function $f(x)$. Then $f(x_0)$ is the absolute maximum value of $f(x)$ on D, if $f(x_0) \geq f(x)$; $\forall x \in D$ and $f(x_0)$ is the absolute minimum value of $f(x)$ on D if $f(x_0) \leq f(x)$; $\forall x \in D$.



Extreme Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Steps to find the absolute extrema of a continuous function $f(x)$ on closed interval $[a, b]$:

Step 1 : Find the critical numbers of $f(x)$ in $[a, b]$

Step 2 : Evaluate $f(x)$ at all the critical numbers and at the endpoints a and b

Step 3 : The largest and the smallest of the values in step 2 is the absolute maximum and absolute minimum of $f(x)$ respectively on the closed interval $[a, b]$.

First Derivative Test

Let $(a, f(a))$ be a critical point of function $f(x)$ which is continuous on an open interval I. If $f(x)$ is differentiable on the interval, except possibly at a, then $f(a)$ can be classified as follows: (when moving across the interval I from left to right)

i. If $f'(x)$ changes from negative to positive at a, then $f(x)$ has a local minimum $f(a)$.
ii. If $f'(a)$ changes from positive to negative at a, then $f(x)$ has a local maximum $f(a)$.
iii. If $f'(x)$ is positive on both sides of a or negative on both sides of a, then $f(a)$ is neither a local minimum nor a local maximum.

Problem: 4

Find the local extrema of the function $f(x) = x^3 - 3x^2 - 9x + 2$ using first derivative test.

Answer:

Given $f(x) = x^3 - 3x^2 - 9x + 2$ ... (1)

Differentiate with respect to $'x'$, we get

$$f'(x) = 3x^2 - 6x - 9$$

$$f'(x) = 3(x^2 - 2x - 3) \qquad \ldots (2)$$

To find critical points use $f'(x) = 0$, we get

$$3(x^2 - 2x - 3) = 0$$

$$x^2 - 2x - 3 = 0$$

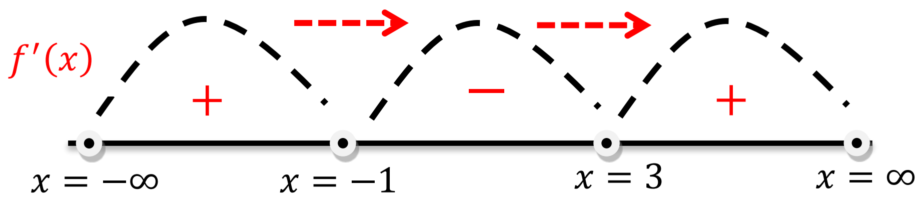
Solving, we get $x = -1; \quad 3$

The critical points are $\boldsymbol{x = -1}$ and $\boldsymbol{x = 3}$.

In the interval $(-\infty, -1)$, then $f'(x) = +\text{ve}$; $\quad [f'(-2) = 3[(-2)^2 - 2(-1) - 3] = 15]$

In the interval $(1,3)$, then $f'(x) = -\text{ve}$; $\quad [f'(2) = 3[(2)^2 - 2(2) - 3] = -9]$

In the interval $(3, \infty)$, then $f'(x) = +\text{ve}$; $\quad [f'(+4) = 3[(4)^2 - 2(4) - 3] = 15]$



From the diagram we see that the value of $f'(x)$ changes its sign from

- positive to negative while passing through $x = -1$, hence it has a local maximum at $x = -1$.
- Negative to positive while passing through $x = 3$, hence it has a local minimum at $x = 3$.

Local maximum value

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2$$
$$= 7$$

Local minimum value

$$f(3) = (3)^3 - 3(3)^2 - 9(3) + 2$$
$$= -25$$

Problem: 5

Find the local extrema of the function $f(x) = 2x^3 + 3x^2 - 12x$ using first derivative test.

Answer:

Given $\quad f(x) = 2x^3 + 3x^2 - 12x \qquad \ldots (1)$

Differentiate with respect to $'x'$, we get

$$f'(x) = 6x^2 + 6x - 12 \qquad \ldots (2)$$

To find critical points use $f'(x) = 0$, we get

$$6x^2 + 6x - 12 = 0$$

$$6(x^2 + x - 2) = 0$$

$$x^2 + x - 2 = 0$$

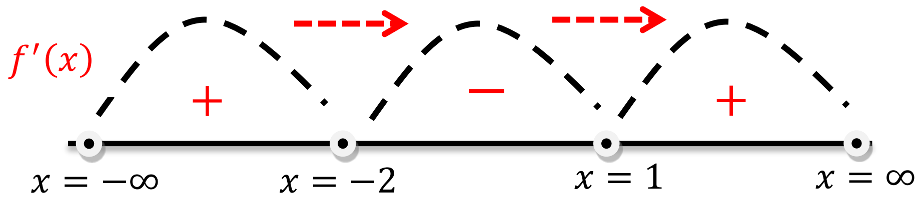
Solving, we get $x = -2;\ 1$

The critical points are $\boldsymbol{x = -2}$ and $\boldsymbol{x = 1}$.

In the interval $(-\infty, -2)$, then $f'(x) = +\text{ve}$; $[f'(-3) = 6(-3)^2 + 6(-3) - 12 = 24]$

In the interval $(-2,1)$, then $f'(x) = -\text{ve}$; $[f'(0) = 6(0)^2 + 6(0) - 12 = -12]$

In the interval $(1, \infty)$, then $f'(x) = +\text{ve}$; $[f'(+3) = 6(3)^2 + 6(3) - 12 = 60]$



From the diagram we see that the value of $f'(x)$ changes its sign from

- Positive to negative while passing through $x = -2$, hence it has a local maximum at $x = -2$.
- Negative to positive while passing through $x = 1$, hence it has a local minimum at $x = 1$.

Local maximum value

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$$
$$= 20$$

Local minimum value

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1)$$
$$= -7$$

Problem: 6

Find the local extrema of the function $f(x) = x^4 - 2x^2 + 3$ using first derivative test.

Answer:

Given $f(x) = x^4 - 2x^2 + 3$ ... (1)

Differentiate with respect to $'x'$, we get

$$f'(x) = 4x^3 - 4x$$

To find critical points use $f'(x) = 0$, we get

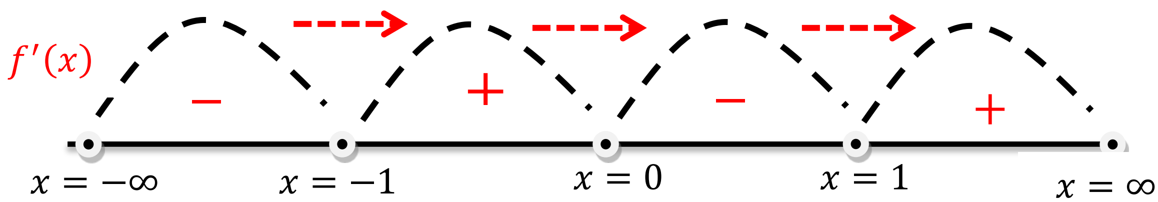
i.e., $\quad 4x[x^2 - 1] = 0$

$4x = 0 \quad \text{or} \quad x^2 - 1 = 0$

$x = 0 \quad \text{or} \quad x^2 = 1$

Solving, we get $\quad x = 0;\ -1;\ 1$

The critical points are $\boldsymbol{x = -1,\ x = 0}$ and $\boldsymbol{x = 1}$.



From the diagram we see that the value of $f'(x)$ changes its sign from

- Negative to positive while passing through $x = -1$ and also at $x = 1$ hence it has a local minimum at $x = -1$ and $x = 1$.
- Positive to negative while passing through $x = 0$, hence it has a local maximum at $x = 0$.

Local minimum value

$$f(1) = (1)^4 - 2(1)^2 + 3$$
$$= 2$$
$$f(-1) = (-1)^4 - 2(-1)^2 + 3$$
$$= 2$$

Local maximum value

$$f(0) = (0)^4 - 2(0)^2 + 3$$
$$= 3$$

Problem: 7

Find the local extrema of the function $f(x) = x^2 e^{-x}$ using first derivative test.

Answer:

Given $\quad f(x) = x^2 e^{-x} \quad \ldots (1)$

Differentiate with respect to $'x'$, we get

$$f'(x) = x^2 \times [e^{-x}(-1)] + e^{-x} \times [2x] \qquad [\because \quad d(uv) = uv' + vu']$$

$$= -x^2e^{-x} + 2xe^{-x}$$

$$f'(x) = xe^{-x}[-x+2]$$

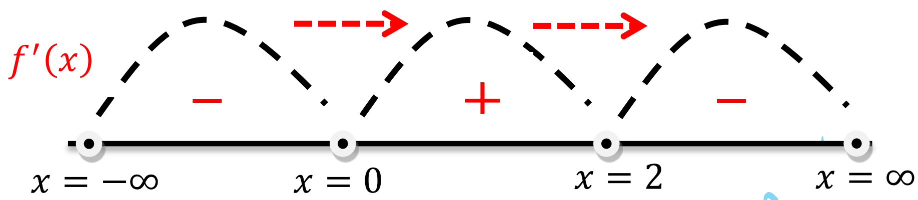
To find critical points use $f'(x) = 0$, we get

i.e., $\quad xe^{-x}[-x+2] = 0$

$x = 0 \quad \text{or} \quad -x+2 = 0$

Solving, we get $\quad x = 0;\ 2$

The critical points are $\boldsymbol{x = 0}$ and $\boldsymbol{x = 2}$.



From the diagram we see that the value of $f'(x)$ changes its sign from

- Negative to positive while passing through $x = 0$, hence it has a local minimum at $x = 0$.
- Positive to negative while passing through $x = 2$, hence it has a local maximum at $x = 2$.

Local minimum value

$$f(0) = 0^2\, e^0$$

$$= 0$$

Local maximum value

$$f(2) = 2^2\, e^{-2}$$

$$= \frac{4}{e^2}$$

$$= 0.5413 \quad \text{(app).}$$

Problem: 8

Find the local extrema of the function $f(x) = \dfrac{x}{1+x^2}$.

Answer:

Given $\quad f(x) = \dfrac{x}{1+x^2}$ $\quad\quad \ldots (1)$

Differentiate with respect to $'x'$, we get

$$f'(x) = \frac{(1+x^2)\times(1) - x\times(0+2x)}{(1+x^2)^2} \qquad \left[\because \; d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}$$

To find critical points use $f'(x) = 0$, we get

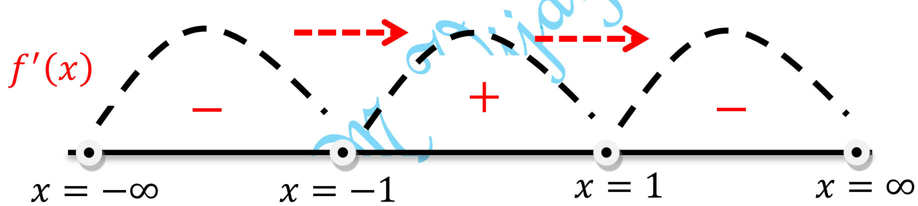
i.e., $$\frac{1-x^2}{(1+x^2)^2} = 0$$

$$1 - x^2 = 0$$

$$x^2 = 1$$

Solving, we get $\quad x = \pm 1$

The critical points are $\boldsymbol{x} = -1$; $\boldsymbol{x} = 1$



From the diagram we see that the value of $f'(x)$ changes its sign from

- Negative to positive while passing through $x = -1$, hence it has a local minimum at $x = -1$.
- Positive to negative while passing through $x = 1$, hence it has a local maximum at $x = 1$.

Local minimum value

$$f(-1) = \frac{-1}{1+(-1)^2} = -\frac{1}{2}$$

Local maximum value

$$f(1) = \frac{1}{1+1^2} = \frac{1}{2}$$

Rolle's Theorem

Suppose that a function $\boldsymbol{f(x)}$ is continuous on the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$ and differentiable on the open interval $(\boldsymbol{a}, \boldsymbol{b})$. Then if $f(\boldsymbol{a}) = \boldsymbol{f}(\boldsymbol{b})$, then there exists at least one point $'\mathbf{c}'$ in the open interval $(\boldsymbol{a}, \boldsymbol{b})$ for which $\boldsymbol{f'}(\boldsymbol{c}) = \mathbf{0}$.

Lagrange's Mean Value Theorem

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one point $x = c$ on this interval, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

It is also as First Mean Value Theorem, and it allows to express the increment of a function on an interval through the value of the derivative at an intermediate point of the segment.

Proof :

Let us take the function

$$F(x) = f(x) + \alpha x \qquad \ldots(1)$$

We choose a number α such that the condition $F(a) = F(b)$ is satisfied. Then

$$f(a) + \alpha a = f(b) + \alpha b$$

$$f(b) - f(a) = \alpha a - \alpha b$$

$$f(b) - f(a) = -\alpha(b - a)$$

$$\alpha = \frac{f(b) - f(a)}{-(b - a)}$$

Using this result in equation (1), we get

$$F(x) = f(x) - \left[\frac{f(b) - f(a)}{(b - a)}\right] x$$

The function $F(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and takes equal values at the endpoints of the interval. Therefore, it satisfies all the conditions of Rolle's theorem. Then there is a point c in the interval (a, b) such that

$$F'(c) = 0$$

This shows that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

or $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remark: 1

Suppose the values of the function $f(x)$ at the endpoints of the interval $[a, b]$ are equal, that is if $f(b) = f(a)$, then the mean value theorem states that there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(b)}{b - a} = 0$$

$$f'(c) = 0$$

we get Rolle's theorem and it is considered as a special case of Lagrange's mean value theorem.

Remark: 2

Suppose if the derivative $f'(x) = 0$ at all points of the interval $[a, b]$, then the function $f(x)$ is constant on this interval. Moreover for any two points say x_1 and x_2 in the interval $[a, b]$, there exists a point $c \in (a, b)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$= 0(x_2 - x_1)$$

$$f(x_2) - f(x_1) = 0$$

$$\therefore \quad f(x_2) = f(x_1)$$

Problem: 1

Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the function $f(x) = x^3 + 2x^2 - x$ in $[-1,2]$.

Answer:

Given $f(x) = x^3 + 2x^2 - x$ and $[a, b] = [-1,2]$

Now $f(a) = f(-1) = (-1)^3 + 2(-1)^2 - (-1) = 2$

and $f(b) = f(2) = (2)^3 + 2(2)^2 - (2) = 14$

Then $f'(x) = 3x^2 + 4x - 1$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 + 4c - 1 = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$3c^2 + 4c - 1 = \frac{14 - 2}{3}$$

$$3c^2 + 4c - 1 = 4$$

$$3c^2 + 4c - 5 = 0$$

Solving, we get $c = 0.78$; $c = -0.21$

$\therefore$ $0.78 \in [-1,2]$ and $-0.21 \notin [-1,2]$

Hence the possible value of $\boldsymbol{c = 0.78}$

Problem: 2

Given $f(x) = x^2 - 3x + 5$. Check Lagrange's mean value theorem holds for this function in the interval $[1,4]$. If yes determine all the numbers c which satisfy the theorem.

Answer:

The given function is continuous and differentiable in $[1,4]$. Therefore we can apply Lagrange's mean value theorem.

Given $f(x) = x^2 - 3x + 5$ and $[a, b] = [1,4]$

Now $f(a) = f(1) = (1)^2 - 3(1) + 5 = 3$

and $f(b) = f(4) = (4)^2 - 3(4) + 5 = 9$

Then $f'(x) = 2x - 3$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c - 3 = \frac{f(4) - f(1)}{4 - 1}$$

$$2c - 3 = \frac{9 - 3}{3}$$

$$2c - 3 = 2$$

$$2c = 5$$

$c = 2.5 \in [1,4]$

Hence the possible value of $\boldsymbol{c = 2.5}$

Problem: 3

Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the function $f(x) = \sqrt{x+4}$ in $[0,5]$.

Answer:

Given $f(x) = \sqrt{x+4}$ and $[a,b] = [0,5]$

Now $f(a) = f(0) = \sqrt{0+4} = 2$

and $f(b) = f(5) = \sqrt{5+4} = 3$

Then $f'(x) = \dfrac{1}{2\sqrt{x+4}} \times \dfrac{d}{dx}(x+4)$

$$= \frac{1}{2\sqrt{x+4}}$$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c+4}} = \frac{f(5) - f(0)}{5-0}$$

$$\frac{1}{2\sqrt{c+4}} = \frac{3-2}{5}$$

$$\frac{1}{\sqrt{c+4}} = 2\left(\frac{1}{5}\right)$$

$$\sqrt{c+4} = \frac{5}{2}$$

Squaring, we get

$$c + 4 = \left(\frac{5}{2}\right)^2$$

$$c = \frac{25}{4} - 4$$

$$= 2.25$$

$\therefore \quad 2.25 \in [0,5]$

Hence the possible value of $\boldsymbol{c = 2.25}$

Problem: 4

Using mean value theorem, prove that $|\sin a - \sin b| \leq |a - b|$, where a and b any real numbers.

Answer:

Given $f(x) = \sin x$ which is a continuous and differentiable in any open interval.

Take the interval [a, b].

Now $f(a) = \sin a$ and $f(b) = \sin b$

Then $f'(x) = \cos x$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\cos x = \frac{\sin b - \sin a}{b - a}$$

Taking modulus on both sides, we get

$$\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos x| \leq 1 \qquad [\because \cos\theta \in [0,1]]$$

$$|\sin b - \sin a| \leq |b - a|$$

Problem: 5

Using mean value theorem, prove that $|\cos a - \cos b| \leq |a - b|$, where a and b any real numbers.

Answer:

Given $f(x) = \cos x$ which is a continuous and differentiable in any open interval.

Take the interval $[a, b]$.

Now $f(a) = \cos a$ and $f(b) = \cos b$

Then $f'(x) = -\sin x$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$-\sin x = \frac{\cos b - \cos a}{b - a}$$

Taking modulus on both sides, we get

$$\left|\frac{\cos b - \cos a}{b - a}\right| = |-\sin x| \le 1 \qquad [\because \ \sin\theta \in [0,1]]$$

$$|\cos b - \cos a| \le |b - a|$$

Problem: 6

Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, also given that $f(-7) = -3$ and $f'(x) \le 2$. What is the largest possible value for $f(0)$?

Answer:

Given $[a,b] = [-7,0]$ and $f(-7) = -3$ also $f'(x) \le 2$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{f(0) - f(-7)}{0 - (-7)} = f'(c)$$

$$f(0) + 3 = 7 \times f'(c)$$

$$f(0) \le 7 \times 2(c) - 3$$

$$\boldsymbol{f(0) \le 11}$$

Problem: 7

Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-2,6]$, also given that $f(-2) = 4$ and $f'(x) \le 3$. What is the largest possible value of upper bound.

Answer:

Given $[a,b] = [-2,6]$ and $f(-2) = 4$ also $f'(x) \le 3$

By definition of Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{f(6) - f(-2)}{6 - (-2)} = f'(c)$$

$$f(6) - 4 = 8 \times f'(c)$$

$$f(6) \le 8 \times (3) + 4$$

$$\boldsymbol{f(6) \le 28}$$